POSITIVE TOPOLOGICAL ENTROPY FOR SEMI-RIEMANNIAN GEODESIC FLOWS

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Abstract. We consider a semi-Riemannian metric whose associated geodesic flow either contains a non-hyperbolic periodic orbit or has infinitely many hyperbolic periodic orbits. Under some conditions, we show that the metric can be C^2-perturbed such that the geodesic flow exhibits positive topological entropy, there are infinitely many non-lightlike closed geodesics, and their number grows exponentially with respect to the length.

Keywords: Semi-Riemannian manifolds, closed geodesics, topological entropy, hyperbolic sets.
MSC2020: Primary: 37D40, 37B40, 53B30 Secondary: 37D30, 58D17

1. Introduction

Semi-Riemannian geometry extends Riemannian geometry by relaxing the requirement that the metric has to be positive definite. In this broader framework, many geometric structures remain valid. However, certain results, such as those related to the geodesic flow, object of this work, do not carry over as easily. For instance, in Riemannian geometry, the Lyusternik-Fet theorem guarantees the existence of closed geodesics on closed manifolds. On surfaces, the number of closed geodesics is infinite, and in higher dimensions, this holds generically, as shown by Rademacher's theorem.

In contrast, the semi-Riemannian case presents a more complex picture (cf. [12]). We mention here some results for the particular case of Lorentzian metrics. For closed orientable Lorentzian surfaces, it has been proven that at least two simple closed geodesics exist, one of which is either timelike or spacelike [28]. There are examples of surfaces that only have those two closed geodesics. For the other surfaces, whether additional closed geodesics exist remains an open question, as does the case for Lorentzian manifolds in dimensions greater or equal than three (see [17, 12] and references therein for some partial results).

In the following, we demonstrate that the existence of even one elliptic closed geodesic is sufficient to find a nearby metric with infinitely many closed geodesics. This results from the presence of a hyperbolic set, which introduces significant dynamical complexity, as indicated by positive topological entropy. A similar conclusion holds under some conditions if the number of hyperbolic closed geodesics is infinite. Note that periodic orbits of the geodesic flow project to closed geodesics on the manifold.

Date: January 16, 2025.

Theorem 1.1. *If the geodesic flow of a C^2 semi-Riemannian complete metric on a smooth closed manifold has a quasi-elliptic periodic orbit, then the metric can be C^∞-perturbed such that it has a non-trivial non-lightlike hyperbolic basic set.*

Starting with a non-hyperbolic non-lightlike periodic orbit, we can use the version of Franks' lemma for semi-Riemannian metrics (see Theorem 3.6) to reduce to the previous case.

Corollary 1.2. *If the metric has a non-hyperbolic non-lightlike periodic orbit, then it can be C^2-perturbed such that it has a non-trivial non-lightlike hyperbolic basic set.*

We now deal with the case of hyperbolic periodic orbits. Notice that if they are lightlike, by hyperbolic stability there are also non-lightlike hyperbolic periodic orbits.

Let $\mathcal{H}$ be the set of C^2 semi-Riemannian complete metrics on a smooth closed manifold whose geodesic flows satisfy the following properties:

(1) all non-lightlike periodic orbits are hyperbolic,
(2) there are infinitely many non-lightlike periodic orbits with infinitely many different periods.

Denote by $\mathcal{F}^*$ the C^2-interior of $\mathcal{H}$.

Theorem 1.3. *For any metric in $\mathcal{F}^*$ the closure of the set of periodic orbits contains a non-trivial non-lightlike hyperbolic basic set.*

A C^2 semi-Riemannian complete metric belongs to $\mathcal{P}$ if its geodesic flow has a quasi-elliptic periodic orbit, a non-hyperbolic non-lightlike periodic orbit or it belongs to $\mathcal{F}^*$. The above results imply that any metric in $\mathcal{P}$ can be perturbed to create a non-trivial non-lightlike hyperbolic basic set.

Recall that a non-trivial hyperbolic basic set for the geodesic flow of a metric g is a hyperbolic, infinite, transitive and locally maximal set (see e.g. [21]), which is C^2-stable. Such a set contains a transverse homoclinic point, thus the topological entropy $h_{top}(g)$ is positive and there are infinitely many periodic orbits. Moreover, the number of periodic orbits grows exponentially with the period (cf. [21, Theorem 18.5.1]). By the relation between the period T of the periodic orbit of the geodesic flow and the length of a closed non-lightlike geodesic $\gamma(t)$, $t \in [0, T]$,

$$\ell_g(\gamma) = \int_0^T \sqrt{|g(\dot\gamma(t), \dot\gamma(t))|}\, dt,$$

we have the following consequence.

Corollary 1.4. *Any metric $g \in \mathcal{P}$ can be C^2-approximated by a C^2 semi-Riemannian metric $\tilde g$ such that*

$$\lim_{T\to+\infty} \frac{1}{T} \log P(T) > 0,$$

where $P(T)$ is the number of non-lightlike closed geodesics γ of $\tilde g$ with $\ell_{\tilde g}(\gamma) \le T$.

The proofs of the theorems above follow the foundational ideas of Contreras and Paternain [16] for geodesic flows on Riemannian surfaces, as well as the significant advancement presented in [14], which extends these results to arbitrary Riemannian manifolds. A similar result was later obtained for billiard maps in bodies [3], and more recently, an analogous version was established for Tonelli Lagrangians [15].

In Section 2, we review some concepts of semi-Riemannian manifolds and their associated geodesic flows. Section 3 discusses four key perturbative results: the bumpy metric theorem, the Klingenberg-Takens theorem, the Kupka-Smale theorem, and Franks' lemma. We conclude by proving Theorem 1.1 in Section 4 and Theorem 1.3 in Section 5.

2. Preliminaries

2.1. **Semi-Riemannian geometry.** Let M be a smooth manifold of dimension m. A *C^r-metric g on M*, $r \in \mathbb{N} := \{1, 2, \dots\}$, is a C^r-tensor field of type $(0, 2)$ such that, for every $x \in M$, the bilinear form $g_x \colon T_xM \times T_xM \to \mathbb{R}$ is symmetric, nondegenerate, and the index of g_x is the same for all x. Recall that the *index* of g_x is the largest dimension of a subspace of T_xM on which g_x is negative definite.

We denote by $\mathcal{SR}^r_\nu(M)$ the set of C^r-metrics on M with index ν. Naturally, we also denote

$$\mathcal{SR}^\infty_\nu(M) = \bigcap_{r \in \mathbb{N}} \mathcal{SR}^r_\nu(M).$$

A *semi-Riemannian manifold* of class C^r is a pair (M, g), where M is a smooth manifold and $g \in \mathcal{SR}^r_\nu(M)$ for some index ν. If $\nu = 0$, the metric is called *Riemannian*, and (M, g) is called a *Riemannian manifold.* If $\nu = 1$, the metric is called *Lorentzian*, and (M, g) is called a *Lorentzian manifold.* More details can be found in the literature e.g. [1, 24].

Notice that $\mathcal{SR}^r_0(M)$ is always non empty as Riemannian metrics always exist. On the other hand, there are topological obstructions whenever $\nu \geq 1$, in particular $\mathcal{SR}^r_\nu(M)$ is empty for some manifolds [**?**, 7]. On the other hand, if $\mathcal{SR}^r_\nu(M) \neq \emptyset$ for some r, then $\mathcal{SR}^\ell_\nu(M) \neq \emptyset$ for any $\ell \in \mathbb{N}$ (cf. [7]).

If $(U, x^1, \dots, x^m)$ is a local chart on M, then g can be written as

$$g = g_{ij} dx^i \otimes dx^j,$$

where $g_{ij} = g(\partial_i, \partial_j) \in C^r(U)$ are the *components of g on U* and ∂_i is a shorthand notation for the coordinate vector fields $\partial/\partial x^i$, and we are using the Einstein's summation convention. Note that, since g is nondegenerate, the matrix $[g_{ij}(x)]$ is invertible for all $x \in U$. If we denote its inverse by $[g^{ij}(x)]$, g^{ij} are also C^r functions on U.

A vector $v \in T_xM$ is said to be

- *timelike* if $g_x(v, v) < 0$;
- *lightlike* if $g_x(v, v) = 0$ and $v \neq 0$;
- *spacelike* if $g_x(v, v) > 0$ or $v = 0$.

A curve $\gamma \colon I \to M$ is called timelike, lightlike or spacelike if $\gamma'(t)$ is respectively timelike, lightlike or spacelike, for all t in an interval I. In particular, a point $(x,v) \in TM$ is called timelike, lightlike or spacelike if v is respectively timelike, lightlike or spacelike.

2.2. **The geodesic flow.** Fix a C^2 semi-Riemannian closed manifold (M,g) with $g \in \mathcal{SR}^r_\nu(M)$ complete. Given a tangent vector $v \in T_xM$ at a point $x \in M$, denote by

$$\gamma_{x,v} \colon \mathbb{R} \to M$$

the geodesic such that $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$, as in the Riemannian case, cf. [24]. The *geodesic flow* of g is the one-parameter family of diffeomorphisms on the tangent bundle

$$\begin{aligned} \varphi^t_g \colon TM &\to TM \\ (x,v) &\mapsto (\gamma_{x,v}(t), \dot\gamma_{x,v}(t)) \,. \end{aligned}$$

Since geodesics travel with constant speed it is enough to consider three energy levels corresponding to positive, zero and negative constant values. So, for $\sigma = +1, 0$ or -1, the σ-tangent bundle is defined by:

$$S^\sigma_g M = \{(x,v) \in TM \colon g_x(v,v) = \sigma\} \,.$$

Clearly, each $S^\sigma_g M$ is preserved by φ^t_g. Moreover, as M is compact so is $S^\sigma_g M$.

By writing the canonical projection $\pi \colon S^\sigma_g M \to M$, geodesics γ on M lift to orbits of the geodesic flow $\pi^{-1}\gamma \subset S^\sigma_g M$.

We say that (x,v) in $S^\sigma_g M$ is a *regular point* if $X_g(x,v) \neq 0$, where $X_g(x,v) := \frac{d}{dt}\varphi^t_g|_{t=0}(x,v)$ stands for the vector field of the geodesic flow at (x,v). Given a regular point (x,v), we say that (x,v) is a *periodic point* of the geodesic flow φ^t_g if $\varphi^t_g(x,v) = (x,v)$ for some positive t. The smallest $\tau > 0$ satisfying the condition above is called the *period* of (x,v). In this case, we say that the orbit of (x,v) is a *periodic orbit* of period τ. The projection on M of a periodic orbit is called a closed geodesic.

Notice that nontrivial closed geodesics on M for g are in one-to-one correspondence with the periodic orbits of φ^t_g except for the case of lightlike geodesics. In fact, one could have a closed geodesic γ, i.e. $\gamma(a) = \gamma(b)$, with $\dot\gamma(a) = \lambda\dot\gamma(b)$ and $\lambda \neq 1$. This is allowed since $g(\dot\gamma,\dot\gamma) = 0$ if the geodesic is lightlike. So, this closed geodesic does not correspond to a periodic orbit. We refer to [2] for more on closed geodesics in the semi-Riemannian setting.

It is widely known that the geodesic flow is the Hamiltonian flow of the Hamiltonian function $(x,v) \mapsto \frac{1}{2}g_x(v,v)$ on TM for a symplectic form depending on g. This is related by the Legendre transform to a similar Hamiltonian function on the cotangent bundle T^*M with the symplectic form which does not depend on the metric.

2.3. **Periodic points.** A *transversal* Σ to the geodesic flow at a regular point (x,v) in $S^\sigma_g M$ is a $(2m-2)$-dimensional smooth symplectic submanifold satisfying

$$T_{(x,v)} S^\sigma_g M = T_{(x,v)}\Sigma + \mathbb{R}X_g(x,v). \tag{2.1}$$

Consider a C^1-family of transversals $\Sigma_t := \Sigma_t(x,v)$ to the flow at $\varphi^t_g(x,v)$, $t \geq 0$, and neighborhoods $U_t \subset S^\sigma_g M$ of (x,v). The *transversal Poincaré* *flow* of g at (x,v) is defined to be the family of C^1-symplectomorphisms

$$P_g^t\colon \Sigma_0 \cap U_t \to \Sigma_t$$

given by $P_g^t(y,u) = \varphi_g^{\Theta(y,u,t)}(y,u)$ with

$$\Theta(y,u,t) = \min\{s \geq 0\colon \varphi_g^s(y,u) \in \Sigma_t\}.$$

We assume that U_t is sufficiently small such that, by the implicit function theorem, Θ is C^1 and $\Theta(U_t,t)$ is bounded for a fixed $t>0$.

The *transversal linear Poincaré flow* of g at (x,v) is the derivative of P_g^t at (x,v),

$$DP_g^t(x,v)\colon T_{(x,v)}\Sigma_0 \to T_{\varphi_g^t(x,v)}\Sigma_t.$$

Given a C^2-metric g and a φ_g^t-invariant, compact and regular set $\Lambda \subset S_g^\sigma M$, we say that Λ is *uniformly hyperbolic* if there exist $\theta \in (0,1)$ and $\eta > 0$ and a DP_g^t-invariant splitting $E_\Lambda^s \oplus E_\Lambda^u$ of $T\Sigma_\Lambda$ such that for any $(x,v) \in \Lambda$ we have

$$\|DP_g^\eta(x,v)|_{E^s_{(x,v)}}\| \leq \theta \quad \text{and} \quad \|DP_g^{-\eta}(\varphi_g^\eta(x,v))|_{E^u_{\varphi_g^\eta(x,v)}}\| \leq \theta.$$

Notice that this notion should refer to the tangent map $D\varphi_g^t$. However, as proven in [4] for Hamiltonian flows (hence also for geodesic flows), it is enough to deal with the associated transversal linear Poincaré flow.

When (x,v) is periodic of period $\tau > 0$ we call P_g^τ the *Poincaré map* and Σ_0 the *Poincaré section*. The periodic point is *degenerate* if $DP_g^\tau(x,v)$ has an eigenvalue which is a root of unity. That is, 1 is an eigenvalue of $DP_g^{k\tau}(x,v)$ for some $k \in \mathbb{Z}$. So, if $e^{2\pi i\lambda}$ is an eigenvalue, then $\lambda \in \mathbb{Q}$. If all eigenvalues of $DP_g^\tau(x,v)$ are ± 1, it is called *parabolic*. A metric is called *bumpy* if all the periodic points are non-degenerate.

It is simple to check that non-degenerate periodic orbits are isolated in the σ-tangent bundle. Moreover, these orbits persist under perturbation of the metric. The same holds for the projected closed geodesics.

The periodic point (x,v) is called *hyperbolic* if its orbit is a uniformly hyperbolic set. This means that all the eigenvalues of $DP_g^\tau(x,v)$ are outside the unit circle. Recall that the eigenvalues are independent of the choice of the transversal and of the point in the periodic orbit.

Finally, the periodic point is *q-elliptic* if it is non-degenerate and non-hyperbolic and $DP_g^\tau(x,v)$ has exactly $2q$ non-real eigenvalues with modulus 1, $1 \leq q \leq m-1$. The remaining eigenvalues have norm different from 1. When $q = m-1$ we call it *totally elliptic*. Notice that, from the above, a bumpy metric only has hyperbolic or q-elliptic periodic orbits.

2.4. **Invariant manifolds.** Consider any distance d on $S_g^\sigma M$ coming from a Riemannian metric. The *stable manifold* of the hyperbolic periodic point (x,v) is the set

$$W^s(x,v) = \{(\tilde{x},\tilde{v})\colon \lim_{t\to+\infty} d(\varphi_g^t(x,v),\varphi_g^t(\tilde{x},\tilde{v})) = 0\}.$$

Similarly, the *unstable manifold* is given by

$$W^u(x,v) = \{(\tilde{x},\tilde{v})\colon \lim_{t\to-\infty} d(\varphi_g^t(x,v),\varphi_g^t(\tilde{x},\tilde{v})) = 0\}. \tag{2.2}$$

We obtain the *weak stable* and *unstable* m-dimensional Lagrangian manifolds of the orbit $\theta := \bigcup_{t\in\mathbb{R}} \varphi_g^t(x,v)$ by taking

$$W^{ws}(\theta) = \bigcup_{t\in\mathbb{R}} \varphi_g^t(W^s(x,v)) \quad \text{and} \quad W^{wu}(\theta) = \bigcup_{t\in\mathbb{R}} \varphi_g^t(W^u(x,v)).$$

Given two hyperbolic periodic points $(x_1,v_1),(x_2,v_2) \in S_g^\sigma M$ with orbits θ_1 and θ_2, as both $W^{ws}(\theta_1)$ and $W^{wu}(\theta_2)$ are contained in a $(2m-1)$-dimensional manifold, we may wonder whether their intersection is transversal.

We say $(x,v) \in W^{ws}(\theta_1)\cap W^{wu}(\theta_2)$ is a *heteroclinic point*. It is *transversal* if

$$T_{(x,v)}W^{ws}(\theta_1) + T_{(x,v)}W^{ws}(\theta_2) = T_{(x,v)}S_g^\sigma M. \tag{2.3}$$

If $\theta_1 = \theta_2$ we call it *homoclinic point*.

Using (2.1) we rewrite (2.3) as

$$T_{(x,v)}W^s(\theta_1) + T_{(x,v)}W^s(\theta_2) + \mathbb{R}X_g(x,v) = T_{(x,v)}S_g^\sigma M.$$

We say that $W^{ws}(\theta_1)$ and $W^{wu}(\theta_2)$ intersect transversally, denoted by $W^{ws}(\theta_1) \pitchfork W^{wu}(\theta_2)$, when all the points in $W^{ws}(\theta_1) \cap W^{wu}(\theta_2)$ are transversal. As common practice, when $W^{ws}(\theta_1) \cap W^{wu}(\theta_2) = \emptyset$ we also say that both submanifolds intersect transversally.

3. Perturbative lemmas for semi-Riemannian metrics

3.1. **C^r-topology.** Given a closed manifold M such that $\mathcal{SR}_\nu^r(M)$ is non-empty, we fix an atlas

$$\{(U_\ell,\varphi_\ell)\colon \ell\in I\}.$$

Recall the Whitney C^r topology, $r\in\mathbb{N}$, given by the norm

$$\|g\|_{C^r} = \max_{0\le r'\le r}\max_{\ell\in I}\max_{x\in\varphi_\ell(U_\ell)}\max_{1\le i,j\le m}\|D^{r'}(g_{ij}\circ\varphi_\ell^{-1})(x)\|$$

for any g in $\mathcal{SR}_\nu^r(M)$, which is therefore a Baire space.

The union of the C^r-open sets of $\mathcal{SR}_\nu^\infty(M)$ for $r\in\mathbb{N}$ form a basis for the Whitney C^∞-topology, making this also a Baire space.

3.2. **Bumpy metric theorem.** Fixing any $\tau > 0$, it is known that for an open and dense subset of semi-Riemannian metrics, the geodesic flow yields only elliptic or hyperbolic periodic ponts with period less than τ (see below). This is a consequence of the semi-Riemannian bumpy metric theorem:

Theorem 3.1. [8, Theorem 3.14] *The set of bumpy semi-Riemannian metrics (i.e. all periodic points are non-degenerate) is generic in $\mathcal{SR}_\nu^r(M)$, $r\ge 2$.*

Corollary 3.2. *There exists a residual set $\mathcal{O}\subset\mathcal{SR}_\nu^r(M)$ such that for $g\in\mathcal{O}$ and any $\tau>0$, the set of periodic orbits of φ_g on $S_g^\sigma M$ with period $\le\tau$ is finite.*

Proof. Suppose there are infinite periodic orbits with period $\le\tau$. Take a point in each of these periodic orbits. Since $S_g^\sigma M$ is compact, there is an accumulation point. This point sits in a periodic orbit because the orbits of the approximating periodic orbits are all bounded by τ. Therefore, there is a non isolated periodic orbit which is degenerate. Theorem 3.1 completes the proof. □

3.3. **Klingenberg-Takens theorem.** The Klingenberg-Takens theorem for semi-Riemannian geodesic flows gives a way to perturb semi-Riemannian metrics in order that the jets of Poincaré maps of closed orbits belong to a given invariant open dense set. This allows us to conclude that generically the 1-jets of Poincaré maps of periodic orbits are hyperbolic or elliptic. Moreover, it allows also the use of the third derivative of the map (3-jet) to establish the weakly monotonous property, crucial to study the local behaviour of weakly monotonous elliptic points. This plays a major role in the proof of Theorem 1.1.

Given $r \in \mathbb{N}$ the set of r-jets at 0 and fixing 0 formed by symplectic automorphisms of $\mathbb{R}^{m-1} \oplus \mathbb{R}^{m-1}$ is denoted by $J^r_s(m-1)$. We say that $Q \subset J^r_s(m-1)$ is *invariant* if $\sigma Q \sigma^{-1} = Q$ for all $\sigma \subset J^r_s(m-1)$ which is invertible. In brief terms linearized Poincaré maps are 1-jets $J^1_s(m-1)$ and also elements of the Lie group $Sp(m-1,\mathbb{R})$.

We present below the semi-Riemannian version of the Klingenberg-Takens theorem:

Theorem 3.3. [8, Corollary 4.2] *Let $r \in \mathbb{N}$ be fixed and let Q be a dense open and invariant subset of $J^r_s(m-1)$. Then, for every $\ell > r$ or $\ell = \infty$, the set $\mathcal{M}_Q$ of all metrics $g \in \mathcal{SR}^\ell_\nu(M)$ such that:*

(i) *all closed geodesics are non-lightlike and non-degenerate,*
(ii) *given any closed geodesic γ, the r^{th} jet of the associated Poincaré map P_g belongs to Q,*

is generic in $\mathcal{SR}^\ell_\nu(M)$.

3.4. **Kupka-Smale theorem.** The classic Kupka-Smale theorem is a generic result displaying two parts. It first states that periodic orbits are hyperbolic or elliptic. Recall that hyperbolic periodic orbits of φ^t_g exhibit a decomposition of the tangent bundle into two Lagrangian m-dimensional subspaces (section 2.4). In addition, it says that all heteroclinic points of hyperbolic periodic orbits are transversal.

The theorem is mainly a result about transversality of submanifolds. The first part follows from Theorem 3.3 since Q can be obtained from the linear symplectic matricial theory which states that hyperbolic and elliptic matrices are open and dense among the symplectic ones. So, we are left to prove the following statement on transversal heteroclinic intersections.

Theorem 3.4. *For a C^r-residual set $\mathcal{R} \subset \mathcal{SR}^r_\nu(M)$, $r \geq 2$ or $r = \infty$, all heteroclinic points of hyperbolic periodic orbits are transversal.*

The rest of this section is dedicated to the proof of Theorem 3.4.

Let $\mathcal{K}^r_N \subset \mathcal{SR}^r_\nu(M)$ be the subset of semi-Riemannian metrics such that given two hyperbolic periodic points γ and η with period $\leq N$, we have

$$W^{ws}_N(\gamma) \pitchfork W^{wu}_N(\eta) \neq \emptyset,$$

where $W^{ws}_N(\gamma)$ is given by those points $\theta \in W^{ws}(\gamma)$ with $d_{W^{ws}(\gamma)}(\theta,\gamma) < N$ (analogous definition for $W^{wu}_N(\eta)$). By Corollary 3.2, generically there are only a finite number of those hyperbolic periodic points.

Since the stable and unstable manifolds of a periodic orbit depend continuously on compact parts in the C^1 topology, we conclude that $\mathcal{K}_N^r$ is an open subset of $\mathcal{SR}_\nu^r(M)$. If we prove that $\mathcal{K}_N^r$ is a dense subset of $\mathcal{SR}_\nu^r(M)$, then the residual subset of Theorem 3.4 is defined by

$$\mathcal{R} = \bigcap_{N\in\mathbb{N}} \mathcal{K}_N^r.$$

The proof that $\mathcal{K}_N^r$ is dense in $\mathcal{SR}_\nu^r(M)$ follows the same lines of [16] where the fundamental key ingredient is the construction of a local perturbation of the metric that guarantees the transversality [16, Lemma 2.6]. Since that lemma cannot be applied to our semi-Riemannian context, we use Lemma 3.5 below instead, which completes the proof of Theorem 3.4.

Lemma 3.5. *Let $g \in \mathcal{SR}_\nu^r(M)$, $r \in \mathbb{N}\cup\{\infty\}$, with hyperbolic periodic orbits γ and η in $S_g^\sigma M$, $\sigma \in \{+1, 0, -1\}$ and $\theta \in W^u(\gamma)$. If the projection π of $W^u(\gamma)$ is a diffeomorphism in a neighborhood of θ, for every sufficiently small neighborhoods $\theta \in V \subset \overline{V} \subset U \subset S_g^\sigma M$ such that $\pi(U)$ does not intersect any closed geodesic of period $\leq N$, we can find $\overline{g} \in \mathcal{SR}_\nu^r(M)$ verifying*

1. *$\overline{g}$ is arbitrarly C^r-close to g,*
2. *$g = \overline{g}$ outside $\pi(U)$,*
3. *γ and η are still periodic orbits for $\overline{g}$,*
4. *the connected component of $W_N^u(\gamma)\cap V$ containing θ and $W^s(\eta)$ are transversal.*

Proof. We follow closely the proof of [16, Lemma 2.6], focusing on the differences coming from the semi-Riemannian setting.

Locally around θ in the unstable manifold of γ, using the Legendre transform, the geodesic flow is the Hamiltonian flow in the form $H(x, y) = \frac{1}{2}g^{ij}(x)y_iy_j$ defined in the cotangent bundle T^*M with the canonical symplectic form $\sum dx_i \wedge dy_i$.

The matrix $G = [g^{ij}]$ is symmetric but might not be positive definite. If so, consider the eigenvalues λ_i which can be positive or negative. Thus, there are symmetric matrices $G_1, G_2 > 0$ such that $G = G_1 - G_2$ (just take $G_1 = G + tI$, $G_2 = tI$, both symmetric, with $t > 0$ such that $G_1 > 0$ because its eigenvalues are $\lambda_i + t$ which are all greater than zero for sufficiently large t). Therefore, $H = H_1 - H_2$ with $H_k(x, y) = \frac{1}{2}y^T G_k(x)y$, $k = 1, 2$, that is positive for $y \neq 0$.

Recall that $\theta \in H^{-1}(\sigma/2)$. As in [16, Lemma 2.6] we want to perturb the metric in $\pi(U)$ so that $W^u(\gamma)$ becomes the graph $\mathcal{G}$ of the one-form $p\colon \pi(U) \to T^*_{\pi(U)}M$ and there is no perturbation to the periodic orbits.

Let $\alpha_k = H_k(\theta)$ and thus $\alpha_1 - \alpha_2 = \sigma/2$. Take $\bar{G} = [\bar{g}^{ij}]$ where

$$\bar{g}^{ij}(x) = \frac{\alpha_1}{H_1(x, p(x))}g_1^{ij}(x) - \frac{\alpha_2}{H_2(x, p(x))}g_2^{ij}(x), \quad x \in \pi(U),$$

and $\bar{g}^{ij}(x) = g^{ij}(x)$ otherwise. Notice that $\bar{g}^{ij}$ is C^r-close to g^{ij} from the closeness of $\mathcal{G}$ and $W^u(\gamma)$, as we only need to estimate $\|H_k(\cdot, p(\cdot)) - \alpha_k\|_{C^r}$. Therefore, by relating the norm of the difference of the inverse matrices, $\bar{g}$ is C^r-close to g. The perturbed Hamiltonian is $\bar{H} = \bar{H}_1 - \bar{H}_2$ with $\bar{H}_k(x, y) = \frac{1}{2}y^T\bar{G}_k(x)y$.

Finally, $\mathcal{G}$ is inside the energy level set since

$$\bar{H}(x,p(x)) = \bar{H}_1(x,p(x)) - \bar{H}_2(x,p(x)) = \alpha_1 - \alpha_2 = \sigma/2.$$

Moreover, $\mathcal{G}$ is a Lagragian submanifold, hence it is invariant by [16, Lemma A.1]. □

3.5. **Franks' lemma.** The original Franks' lemma asserts that perturbations of the derivative of a diffeomorphism at a finite set can be realized as derivatives of a C^1-close diffeomorphism. The C^1 setting is crucial, as the result no longer holds in the C^2 topology [25]. First proved by John Franks in [18, Lemma 1.1], this lemma has become an essential tool for establishing many fundamental results in the stability and genericity theories of dynamical systems.

The first version of Franks' lemma for geodesic flows appeared in [14], with subsequent extensions provided in [23]. In the context of geodesic flows, the perturbations are made to the metric, which are inherently non-local in phase space. This poses a significant challenge, making these adaptations considerably more difficult to achieve. Similar difficulties arise in the versions for planar billiards [29] and for billiards in bodies [3].

In this section, we present Franks' lemma for semi-Riemannian metrics, whose proof follows a straightforward adaptation of [23, Theorem 1.1], originally formulated for the Riemannian case.

Let $\mathfrak{g}$ be any Riemannian metric of M. We use it to define a tubular neighborhood in M with radius $\rho > 0$ of a curve Γ:

$$\mathcal{C}_{\mathfrak{g}}(\Gamma, \rho) = \{x \in M \colon d_{\mathfrak{g}}(x, \Gamma) < \rho\}.$$

Notice that in [23] $\mathfrak{g} = g$ since both are Riemannian, whilst in our present setting we need to distinguish them.

Theorem 3.6 (Franks' lemma for semi-Riemannian metrics)**.** *Let (M, g) be a smooth compact semi-Riemannian manifold of dimension ≥ 2. For every $T > 0$ there exists $\delta_T, \tau_T, K_T > 0$ such that the following holds. For every non-lightlike geodesic $\gamma_\theta \colon [0,T] \to M$ we can find $\bar{t} \in [0, T - \tau_T]$ and $\overline{\rho} > 0$ with*

$$\mathcal{C}_{\mathfrak{g}}\big(\gamma_\theta([\bar{t}, \bar{t} + \tau_T]), \overline{\rho}\big) \cap \gamma_\theta([0,T]) = \gamma_\theta((\bar{t}, \bar{t} + \tau_T)),$$

such that for every $0 < \delta < \delta_T$ for each $A \in \mathrm{Sp}(m-1)$ satisfying

$$\|A - P_g(\gamma)(T)\| < \delta$$

and for every $0 < \rho < \overline{\rho}$ there exists a C^∞ semi-Riemannian metric h on M that is conformal to g in the form $h = e^\beta g$ with $\beta \in C^\infty(M)$, such that:

1. *$\gamma_\theta \colon [0,T] \to M$ is still a non-lightlike geodesic of (M, h),*
2. *$Supp(\beta) \subset \mathcal{C}_{\mathfrak{g}}\big(\gamma_\theta([\bar{t}, \bar{t} + \tau_T]), \rho\big)$,*
3. *$\|e^\beta - 1\|_{C^2} < K_T\sqrt{\delta}$,*
4. *$P_h(\gamma_\theta)(T) = A$.*

The proof in [23] relies on an abstract control theory result [23, Proposition 2.4] that is applied to the Jacobi equation. For semi-Riemannian metrics we also obtain the Jacobi equation since all the involved geometric structures are also valid in this more general context (cf. [24]). On the other hand, the above result is restricted to non-lightlike geodesics, because Fermi coordinates are only available for such geodesics.

4. Proof of Theorem 1.1

In this section we show that if there is an elliptic periodic orbit, then there is a C^∞-perturbation of the metric so that the geodesic flow has a horseshoe. This is done by considering Kupka-Smale metrics and reducing the dimension of the problem using a general result by Contreras, Herman and Arnaud for symplectic twist maps associated to the Poincaré map. This shows the existence of a transversal heteroclinic orbit. We follow closely [14] as the adaptation only requires the use of the version of the Klingenberg-Takens theorem for semi-Riemannian metrics (Theorem 3.3).

Take a q-elliptic periodic point. If it is lightlike, by persistency of elliptic periodic points, there is a nearby non-lightlike q-elliptic periodic point (x, v). Let P_g^t be the transversal Poincaré flow of $g \in \mathcal{SR}_\nu^r(M)$, $r \in \{2, 3, \dots, \infty\}$, at (x, v). We restrict it to the center manifold $W^c(x, v)$ in a small enough neighborhood of (x, v). In appropriate coordinates, it is a C^{r-1}-diffeomorphism $f\colon \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ preserving the canonical symplectic form ω_0 and fixing the origin. If the origin is 4-elementary, then we say that (x, v) is a 4-elementary totally elliptic fixed point. So, using the Birkhoff normal form, if (x, v) is weakly monotonous, then this symplectomorphism can be conjugated to a weakly monotonous (twist) map on $\mathbb{T}^q \times \mathbb{R}^q$. Notice that this is related to the 3-jet of the map, so by Theorem 3.3 we obtain the following.

Proposition 4.1. *There is a C^∞-residual set $\mathcal{R}_1 \subset \mathcal{SR}_\nu^2(M)$ such that for any $g \in \mathcal{R}_1$ any periodic orbit is either hyperbolic or 4-elementary weakly monotonous nonlightlike q-elliptic for some $1 \leq q \leq d$.*

We now find conditions for the existence of a 1-elliptic periodic point nearby a q-elliptic one. This follows from the fact that the Birkhoff normal form can be put in new coordinates so that it is C^1-close to a weakly monotonous completely integrable exact symplectomorphism on a strip around $\mathbb{T}^q \times \{0\}$ (cf. [14]). We can now apply the following result.

Theorem 4.2 ([14, Theorem 4.1])**.** *If $F\colon \mathbb{T}^q \times \mathbb{R}^q \to \mathbb{T}^q \times \mathbb{R}^q$ is a Kupka-Smale weakly monotonous exact C^4-symplectomorphism C^1-close to a completely integrable symplectomorphism, then it has a 1-elliptic periodic point and a nontrivial hyperbolic set near $\mathbb{T}^q \times \{0\}$.*

Finally, the above discussion completes the goal of this section stated in the next theorem, and also Theorem 1.1 follows.

Theorem 4.3. *Let $g \in \mathcal{SR}_\nu^5(M)$ be Kupka-Smale. There is a C^∞-perturbation such that it has a 1-elliptic periodic point and there is a nontrivial hyperbolic set.*

5. Proof of Theorem 1.3

The set of the non-lightlike periodic orbits of the geodesic flow φ_g^t is denoted by $\mathrm{Per}_\pm(g)$. Let $g \in \mathcal{F}^*$ so that the set $\mathrm{Per}_\pm(g)$ contains only hyperbolic periodic orbits (see condition (1) of the definition of $\mathcal{F}^*$ in Section 1).

We proceed along the same lines as in [6, 10, 27, 20, 5]. Following [11, Corollary 2.18] (see also [13]) and the fact that there are infinitely many different periods (see condition (2) of the definition of $\mathcal{F}^*$), we get the following dichotomy: either there is a uniform dominated splitting on $\mathrm{Per}_\pm(g)$, or by Franks' lemma (Theorem 3.6) there is a semi-Riemannian metric with a non-hyperbolic and non-lightlike periodic orbit. Since this is not allowed in $\mathcal{F}^*$, $\mathrm{Per}_{\pm}(g)$ exhibits a uniform dominated splitting. Recall that in the symplectic case this implies that $\mathrm{Per}_{\pm}(g)$ is partially hyperbolic (see e.g. [9, Theorem 11]).

By restricting the tangent map to the central subspace of the splitting and employing a Jordan normal form for symplectic matrices [22, 19], we once again achieve a partially hyperbolic splitting according to the aforementioned dichotomy. This results in an increase in the dimension of the stable and unstable subspaces of the original tangent map on $\mathrm{Per}_{\pm}(g)$. By repeating this procedure, we conclude that $\mathrm{Per}_{\pm}(g)$ is indeed hyperbolic, and the same applies to its closure because of partial hyperbolicity.

According to Smale's spectral decomposition theorem (see e.g. [26, page 385]), the closure of the set of periodic orbits Λ is a union of finitely many pairwise disjoint basic hyperbolic sets. Since Λ is infinite, at least one of the basic sets must be nontrivial.

Acknowledgements

MB was partially funded by the projects "Means and Extremes in Dynamical Systems" PTDC/MAT-PUR/4048/2021 and UIDB/00144/2020, JLD by the project UIDB/05069/2020, and MJT by the projects UIDB/00013/2020 (https://doi.org/10.54499/UIDB/00013/2020) and UIDP/00013/2020 (https://doi.org/10.54499/UII All these projects were financed by FCT, I.P., the Portuguese national funding agency for science, research and technology. PM was partially funded by project FilmEU+ Ref. 101124314 ERASMUS-EDU-2023-EUR-UNIV. This project is financed by the Erasmus+ Programme of the European Union.

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