POSITIVE LYAPUNOV EXPONENTS VERSUS INTEGRABILITY IN RANDOM CONSERVATIVE DYNAMICS

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ABSTRACT. We study random dynamical systems generated by volume-preserving piecewise C^1 maps. For this class of systems, we establish an invariance principle stating that if all Lyapunov exponents vanish, then there exists a measurable family of probability measures on the projective bundle that is invariant under the projective cocycle induced by the derivative. We apply this principle to two classes of random systems. First, we consider random additive perturbations of the billiard map associated with a strictly convex planar table on a surface of constant curvature. In this setting, we show that the Lyapunov exponents vanish almost everywhere if and only if the billiard table is a geodesic disk. Second, we study random additive perturbations of a standard map and prove that the Lyapunov exponents vanish almost everywhere if and only if the map is integrable.

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1. Introduction

Numerical simulations indicate that generic area-preserving surface maps exhibit a mixed phase space, with coexisting regions of regular and chaotic dynamics. This behavior is observed in a broad class of systems, including area-preserving twist maps and billiards in convex tables, and suggests positivity of the metric entropy with respect to the Lebesgue measure. Despite substantial numerical and heuristic evidence, no general method is currently available to establish this property rigorously.

In this paper, we study random additive perturbations of area-preserving maps on the 2-torus $\mathbb{T}^2$ and investigate whether randomness leads to non-vanishing Lyapunov exponents almost everywhere. More precisely, given an area-preserving map $f : \mathbb{T}^2 \to \mathbb{T}^2$, for each $x \in \mathbb{T}^2$, we define the corresponding additive perturbation by

$$f_x(y) = f(y) + x \bmod 1.$$

We then consider random compositions generated by an i.i.d. sequence $(X_n)_{n\geq 0}$ of $\mathbb{T}^2$-valued random variables whose support contains a neighborhood of the origin. The resulting random dynamical system is given by

$$\mathsf{T}^n = f_{X_{n-1}} \circ \cdots \circ f_{X_0}.$$

Since each map f_x is area-preserving, this construction yields an area-preserving random dynamical system on $\mathbb{T}^2$.

The introduction of additive randomness is expected to weaken invariant structures and enhance ergodic properties, increasing the likelihood of exponential instability and positive metric entropy. This leads to the following question: do conservative random perturbations typically generate non-vanishing Lyapunov exponents almost everywhere, and if not, does the persistence of vanishing Lyapunov exponents characterize integrability of the underlying deterministic system?

We address this question for two fundamental classes of conservative dynamical systems: billiards in convex tables and standard maps.

We begin with billiards on surfaces of constant curvature. Let S be a Riemannian surface of constant curvature, and let $D \subset S$ be a strictly convex domain whose boundary ∂D is a C^2 simple closed curve with positive geodesic curvature. We refer to such a domain D as a *convex table*. A billiard in D describes the motion of a point particle traveling along geodesics of S inside D and undergoing elastic reflections at ∂D.

The dynamics is encoded by the *billiard map* $\Phi : V \to V$, a twist map defined on the collision space $V = S^1 \times [-1, 1]$. Let L denote the length of ∂D. Each point $(s, r) \in V$ uniquely represents a collision with the boundary: the coordinate $s \in \mathbb{R}/(L\mathbb{Z}) \cong S^1$ is the arc-length parameter of the collision point on ∂D, while $r = -\cos\theta \in [-1, 1]$, where $\theta \in [0, \pi]$ is the reflection angle measured with respect to the tangent to ∂D at s. The map Φ sends each collision (s, r) to the next collision along the trajectory. Upon identifying V with the two-torus $\mathbb{T}^2$, the billiard map Φ induces a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ that preserves the Lebesgue measure. Full details of this construction are given in Section 3.

Despite extensive study, it remains an open problem whether billiards in strictly convex tables with smooth boundary can exhibit positive metric entropy with respect to the Lebesgue measure. All known examples of billiards with this property involve tables that are either not strictly convex or not smooth; see, for instance, [12, 35, 18, 30, 25]. It is worth mentioning that, while generic convex billiards have positive topological entropy [7, 14, 37], this fact alone does not yield examples with positive metric entropy with respect to the Lebesgue measure.

Our first main result characterizes precisely when Lyapunov exponents vanish for random billiards, where f denotes the billiard map on $\mathbb{T}^2$, f_x its additive perturbations, and $\{\mathsf{T}^n\}$ the associated random dynamical system. Additive perturbations f_x of the billiard map f admit a clear geometric interpretation. Given $x = (\bar{s}, \bar{r})$ and an initial collision (s_0, r_0), one first computes the next collision $(s_1, r_1) = f(s_0, r_0)$, and then translates the collision point by x in the phase space, shifting the arc-length coordinate by $\bar{s}$ modulo L and the r-coordinate by $\bar{r}$ modulo 2.

Theorem 1.1. *Let D be a convex billiard table on a surface S of constant curvature. Let $(X_n)_{n\geq 0}$ be an i.i.d. sequence of $\mathbb{T}^2$-valued random variables with a distribution that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$ and has support containing a neighbourhood of the origin. Then the Lyapunov exponents of the associated random dynamical system $\{\mathsf{T}^n\}$ vanish almost everywhere if and only if D is a geodesic disk.*

This theorem reflects the exceptional symmetry of circular billiards on surfaces of constant curvature. For such billiards, the phase space of the map f is foliated by non-contractible invariant circles [8, 19], a property often referred to as *total integrability*. When f is replaced by an additive perturbation f_x, this foliation remains globally invariant: although individual circles may no longer be invariant under f_x, each circle of the foliation is mapped by f_x onto another circle. As a consequence, the associated random dynamical system exhibits no exponential growth of derivatives, and hence has vanishing Lyapunov exponents almost everywhere.

Theorem 1.1 extends significantly the work of Nguyen, who showed that for billiards in non-circular elliptical tables in the Euclidean plane, random additive perturbations have non-vanishing Lyapunov exponents almost everywhere [31].

The second class of conservative dynamical systems considered in this paper consists of standard maps and their generalizations. The *standard map*, introduced by Chirikov [16], is a family of diffeomorphisms $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ depending on a real parameter $K \in \mathbb{R}$, defined by

$$f(y_1, y_2) = \big(y_1 + y_2 + K\sin(2\pi y_1),\, y_2 + K\sin(2\pi y_1)\big) \bmod 1$$

for every $(y_1, y_2) \in \mathbb{T}^2$. The case $K = 0$ corresponds to an integrable map.

Standard maps arise naturally in Hamiltonian dynamics as models of periodically kicked systems and play a central role in the study of the transition from integrability to chaos. However, proving the existence of non-vanishing Lyapunov exponents for non-integrable standard maps and their generalizations remains notoriously difficult.

Our second main result establishes a rigidity theorem for random additive perturbations of standard maps, analogous to Theorem 1.1. In this setting, the conclusion holds under weaker assumptions on the common distribution of the random variables $(X_n)_{n\geq 0}$. We say that a probability measure ν on $\mathbb{T}^2$ is *non-degenerate* if it satisfies one of the following conditions:

(i) ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$ and its support contains a neighbourhood of the origin;
(ii) ν is singular with respect to the Lebesgue measure on $\mathbb{T}^2$ but absolutely continuous with respect to the Lebesgue measure dy_2 on the second coordinate, and its support contains an interval of the form $\{0\} \times [-\epsilon, \epsilon]$ for some $\epsilon > 0$.

Theorem 1.2. *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a standard map, and let $(X_n)_{n\geq 0}$ be an i.i.d. sequence of $\mathbb{T}^2$-valued random variables with a non-degenerate distribution. Then the Lyapunov exponents of the associated random dynamical system $\{T^n\}$ vanish almost everywhere if and only if f is integrable, that is, if and only if $K = 0$.*

For random standard maps, Blumenthal, Xue, and Young obtained quantitative lower bounds on the maximal Lyapunov exponent under sufficiently strong random additive perturbations [10]. In contrast, the present work does not rely on large randomness and instead provides a sharp characterization of the vanishing of Lyapunov exponents.

Taken together, Theorems 1.1 and 1.2 show that for random conservative perturbations of billiards and standard maps, vanishing Lyapunov exponents occur if and only if the underlying deterministic system is totally integrable.

The conceptual core of the paper is a rigidity result for random dynamical systems, which we call an invariance principle, following the terminology of Avila–Viana [2]. Roughly speaking, we show that if the Lyapunov exponents of a random system coincide almost everywhere, then there exists a non-random invariant structure on the projective tangent bundle. This is stated precisely in Theorem 2.10 and is derived by applying a theorem of Ledrappier [29] to the setting of random maps considered in this paper. The argument is rooted in Furstenberg's theory of random matrix products [23] and its subsequent extensions to stationary sequences of matrices by Ledrappier [29] and to random diffeomorphisms on manifolds by Carverhill [13] and Baxendale [3].

The paper is organized as follows. Section 2 establishes the general invariance principle for random maps and concludes with results on ergodic properties of the random dynamical systems considered here. Section 3 provides background on convex billiards on surfaces of constant curvature. The proof of Theorem 1.1 is given in Section 3.3. Finally, Section 4 is devoted to random standard maps and contains the proof of Theorem 1.2.

2. INVARIANT PRINCIPLE FOR RANDOM MAPS

2.1. **Random dynamical systems.** We model a random dynamical system as a skew-product over a shift transformation. The definition of a random dynamical system provided below is not the most general. For more general definitions, see [1, 28].

Let M be a complete smooth Riemannian manifold of dimension d equipped with the Borel σ-algebra $\mathcal{B}$. Denote by vol the Riemannian volume measure of M. Throughout this paper, $\mathbb{N}$ will denote the set of natural numbers, including zero.

Let $(X, \mathcal{X}, \nu)$ be a probability space, where X is a separable complete metric space, $\mathcal{X}$ is its Borel σ-algebra, and ν is a probability measure on $\mathcal{X}$. Denote by $(\Omega, \Sigma, \rho_\nu)$ the probability space given by the product of countably many copies of $(X, \mathcal{X}, \nu)$. Namely, Ω is the space of sequences

$$\Omega = X^{\mathbb{N}} = \{(\omega_n)_{n\in\mathbb{N}} \colon \omega_n \in X \text{ for all } n \in \mathbb{N}\},$$

equipped with the product σ-algebra $\Sigma = \mathcal{X}^{\mathbb{N}}$ and the product measure $\rho_\nu = \nu^{\mathbb{N}}$. The shift map $\sigma \colon \Omega \to \Omega$ defined by

$$(\sigma(\omega))_n = \omega_{n+1} \quad \text{for every } \omega \in \Omega,$$

is measurable and preserves the probability measure ρ_ν.

Since X and M are separable complete metric spaces, the same holds for $X \times M$, Ω, and $\Omega \times M$. Each of these spaces is endowed with its Borel σ-algebra.

Let $f\colon X \times M \to M$ be a measurable map. For every $x \in X$, define

$$f_x(\cdot) = f(x,\cdot)\colon M \to M.$$

Definition 2.1. We call the skew-product $F\colon \Omega \times M \to \Omega \times M$ defined by

$$F(\omega, y) = (\sigma(\omega), f_{\omega_0}(y)) \quad \text{for every } (\omega, y) \in \Omega \times M,$$

the *random dynamical system on M generated by f and ν*.

Note that F depends on ω only through its 0th component ω_0. For every $n \in \mathbb{N}$ and every $\omega \in \Omega$, define $F^n_\omega\colon M \to M$ as follows:

$$F^n_\omega = \begin{cases} \mathrm{id}_M & \text{if } n = 0, \\ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0} & \text{if } n \geq 1. \end{cases}$$

Then the iterates of F can be written as

$$F^n(\omega, y) = (\sigma^n(\omega), F^n_\omega(y)) \quad \text{for every } (\omega, y) \in \Omega \times M.$$

Definition 2.2. Let $y \in M$. For every $\omega \in \Omega$, the sequence $\{F^n_\omega(y)\}_{n\geq 0}$ is called the *random orbit of y associated with ω*.

We now formulate the standing assumptions on f and ν that will be assumed throughout this paper.

Assumption A. *There exists a Borel probability measure m on M absolutely continuous with respect to* vol *such that for ν-a.e. $x \in X$,*

(1) *m is invariant with respect to f_x,*
(2) *there exists an open set $N_x \subset M$ with $m(M \setminus N_x) = 0$ such that $f_x|_{N_x}\colon N_x \to M$ is a C^1 embedding.*

Remark 2.3. Assumption A allows the maps f_x to have singularities, as in the case of billiard maps. The singular set of f_x consists of the points where f_x fails to be continuous or differentiable, and is contained in $M \setminus N_x$.

We fix a global measurable trivialization $\Psi\colon TM \to M \times \mathbb{R}^d$ of the tangent bundle TM such that for each $y \in M$, the restriction

$$\Psi_y := \Psi|_{T_yM} : T_yM \to \mathbb{R}^d$$

is a linear isometry between T_yM and $\mathbb{R}^d$ endowed with its standard Euclidean inner product (see [1, Lemma 4.2.4]).

Define a measurable map $A\colon X \times M \to \mathrm{GL}(d, \mathbb{R})$ as follows. For every $x \in X$ and every $y \in N_x$, let

$$A(x, y) = \Psi_{f_x(y)} \circ Df_x(y) \circ \Psi_y^{-1},$$

i.e. $A(x, y)$ is the matrix representation of $Df_x(y)$ with respect to the global trivialization Ψ. For every $x \in X$ and every $y \in M \setminus N_x$, let

$$A(x, y) = I,$$

where I denotes the indentity matrix in $\mathrm{GL}(d, \mathbb{R})$. The norm $\|A(x, y)\|$ is the operator norm of $A(x, y)$ induced by the standard Euclidean inner product on $\mathbb{R}^d$.

Assumption B. *The functions*

$$(x, y) \mapsto \log^+ \|Df_x(y)\|, \quad (x, y) \mapsto \log^+ \|Df_x(y)^{-1}\|$$

belong to $L^1(\nu \times m)$*.*

Remark 2.4. As a consequence of the previous assumption, the probability measure $\rho_\nu \times m$ on $\Omega \times M$ is invariant with respect to F, and the functions $(\omega, y) \mapsto \log^+ \|DF^1_\omega(y)\|$ and $(\omega, y) \mapsto \log^+ \|(DF^1_\omega(y))^{-1}\|$ belong to $L^1(\rho_\nu \times m)$.

Definition 2.5. The maximal and minimal Lyapunov exponents of F at $(\omega, y) \in \Omega \times M$ are defined, respectively, by

$$\lambda^+_F(\omega, y) = \limsup_{n \to +\infty} \frac{1}{n} \log \|DF^n_\omega(y)\|$$

and

$$\lambda^-_F(\omega, y) = -\limsup_{n \to +\infty} \frac{1}{n} \log \left\|(DF^n_\omega(y))^{-1}\right\|.$$

Remark 2.6. By the Subadditive Ergodic Theorem the limsup in the definitions of $\lambda^+_F(\omega, y)$ and $\lambda^-_F(\omega, y)$ can be replaced by the lim for $(\rho_\nu \times m)$-a.e. $(\omega, y) \in \Omega \times M$ [33].

2.2. **Random additive perturbations of toral maps.** We now specialize to random dynamical systems for which both the base space X and the fiber M are the d-dimensional torus,

$$X = M = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d.$$

Moreover, we assume that the map $f \colon X \times \mathbb{T}^d \to \mathbb{T}^d$ is defined by

$$f(x, y) = \tau_x \circ g(y) = g(y) + x \quad \text{for all } x, y \in \mathbb{T}^d,$$

where τ_x denotes the translation on $\mathbb{T}^d$ by x, and $g \colon \mathbb{T}^d \to \mathbb{T}^d$ is a measurable map with the following properties:

(1) g preserves the Riemannian volume vol of $\mathbb{T}^d$,
(2) there exists an open subset $N \subset \mathbb{T}^d$ with $\mathrm{vol}(\mathbb{T}^d \setminus N) = 0$ such that the restriction $g|_N \colon N \to \mathbb{T}^d$ is a C^1 embedding,
(3) the functions $y \mapsto \log^+ \|(Dg(y))^{\pm 1}\|$ belong to $L^1(\mathrm{vol})$.

We take as the probability measure m on $\mathbb{T}^d$ the normalized Riemannian volume vol, i.e. $m = (\mathrm{vol}(\mathbb{T}^d))^{-1}\,\mathrm{vol}$. Note that ν and ρ_ν are now probability measures on $\mathbb{T}^d$ and $\Omega = (\mathbb{T}^d)^{\mathbb{N}}$, respectively.

Definition 2.7. We refer to a map f defined in this way as an *additive perturbation* of the toral map g. A random dynamical system F generated by such an f and probability measure ν on $\mathbb{T}^d$ is called a *ν-random additive perturbation of the toral map g*.

We denote by $\mathrm{SL}(d,\mathbb{R})$ the set of all $d \times d$ real matrices with determinant ± 1.

Remark 2.8. Under the assumptions on g, any additive perturbation F of g satisfies Assumptions A and B. By Oseledets' Theorem [33], the limit

$$\lim_{n\to+\infty} \frac{1}{n} \log |\det DF^n_\omega(y)|$$

exists for $(\rho_\nu \times m)$-almost every $(\omega, y) \in \Omega \times \mathbb{T}^d$ and is equal to the sum of the Lyapunov exponents of F. Since g is volume-preserving, we have

$$Df_x(y) = Dg(y) \in \mathrm{SL}(d,\mathbb{R}) \quad \text{for all } (x,y) \in \mathbb{T}^d \times \mathbb{T}^d.$$

It follows that the limit above is equal to 0. Consequently, for any additive random perturbation F, the condition $\lambda^+_F = \lambda^-_F$ a.e. on $\Omega \times \mathbb{T}^d$ in Theorem 2.10 is equivalent to $\lambda^+_F = 0$ a.e. on $\Omega \times \mathbb{T}^d$.

2.3. **Invariance principle for random maps.** In this section, we first recall a theorem of Ledrappier that generalizes Furstenberg's classical result on zero Lyapunov exponents for products of independent unimodular matrices [23] to linear cocycles over Markov chains. Similar results have been obtained in [32, 34, 24]. Then we apply Ledrappier's theorem to random maps.

2.3.1. *Ledrappier's Theorem.* Let Z be a separable complete metric space with its Borel σ-algebra $\mathcal{Z}$. Consider a Markov chain $\{Z_n\}_{n\geq\mathbb{N}}$ with state space $(Z,\mathcal{Z})$, transition probability $\{P_z : z \in Z\}$ and stationary probability measure ζ.

Let (Ω',Σ') be the two-sided product space $\Omega' = Z^{\mathbb{Z}}$ endowed with the product σ-algebra $\Sigma' := \mathcal{Z}^{\mathbb{Z}}$. For each $n \in \mathbb{Z}$, let $\pi_n : \Omega' \to Z$ be the projection given by $\pi_n(\omega') = \omega'_n$ for all $\omega' = (\omega'_n)_{n\in\mathbb{Z}} \in \Omega'$. Since Z is a separable complete metric space, there exists a probability measure $\mathbb{P}_\zeta$ on Σ' such that the coordinate process $\{\pi_n\}_{n\in\mathbb{Z}}$ is a Markov chain on $(\Omega',\Sigma',\mathbb{P}_\zeta)$ with transition probability P_z and stationary probability measure ζ. Moreover, the shift map $\sigma' : \Omega' \to \Omega'$ is measurable with respect to Σ' and preserves $\mathbb{P}_\zeta$ (see [21, Theorem 3.1.7 and Example 5.1.3]).

Let $A : Z \to \mathrm{GL}(d,\mathbb{R})$ be a measurable map such $\log^+ \|A(\cdot)^{\pm1}\|$ belong to $L^1(\zeta)$. Consider the liner cocycle over $(\Omega',\Sigma',\mathbb{P}_\zeta,\sigma')$ generated by A, and denote by $\Lambda^+_A(\omega')$ and $\Lambda^-_A(\omega')$ its maximal and minimal Lyapunov exponents, respectively [29].

By abuse of notation, we also denote by $A(z)$ the action induced by the matrix $A(z)$ on the projective space $\mathbb{P}^{d-1}$.

A family of probability measures $\{\eta_z : z \in Z\}$ on $\mathbb{P}^{d-1}$ is said to be *measurable* if for every Borel set $B \subset \mathbb{P}^{d-1}$, the map $z \mapsto \eta_z(B)$ is $\mathcal{Z}$-measurable. Since $\mathbb{P}^{d-1}$ is a compact metrizable space, this condition is equivalent to requiring that the map $z \mapsto \eta_z$ is measurable as a map from $(Z, \mathcal{Z})$ into the space of probability measures on $\mathbb{P}^{d-1}$ endowed with the Borel σ-algebra of the weak-* topology.

Theorem 2.9 ([29, Corollary 2])**.** *Suppose that*

$$\Lambda_A^+(\omega') = \Lambda_A^-(\omega') \quad \text{for } \mathbb{P}_\zeta\text{-a.e. } \omega' \in \Omega'.$$

Then there exists a measurable family $\{\eta_z : z \in Z\}$ of probability measures on $\mathbb{P}^{d-1}$ such that for ζ-a.e. $z \in Z$,

$$A(z)_*\eta_z = \eta_{z_1} \quad \text{for } P_z\text{-a.e. } z_1 \in Z.$$

2.3.2. *Invariant principle for random maps.* We now establish a result (Theorem 2.10) that may be regarded as a generalization of Furstenberg's theorem to random maps satisfying Assumptions A and B. Related results for random diffeomorphisms on manifolds were previously obtained by Carverhill [13] and Baxendale [3].

Denote by $PM = \bigsqcup_{y \in M} P_y M$ the projective bundle of the manifold M, where $P_y M$ is the projective space of $T_x M$. Let $\pi\colon PM \to M$ be the bundle projection.

Any probability measure η on PM such that $\pi_*\eta = m$ admits a *disintegration* with respect to m (see [33]), i.e. a measurable family $\{\eta_y \colon y \in M\}$ of probability measures on PM such that each η_y is concentrated on the fiber $P_y M$ and

$$\eta(E) = \int_M \eta_y(E \cap P_y M)\, dm(y) \quad \text{for every measurable } E \subset PM.$$

We use the same notation $Df_x(y)$ for the differential of f_x at y and for the map it induces on the projective fiber $P_y M$.

Theorem 2.10. *Let $F\colon \Omega \times M \to \Omega \times M$ be a random dynamical system generated by f and ν that satisfies Assumptions A and B. Suppose that*

$$\lambda_F^+(\omega, y) = \lambda_F^-(\omega, y) \quad \text{for } \rho_\nu \times m\text{-a.e. } (\omega, y) \in \Omega \times M.$$

Then there exists a probability measure η on PM with $\pi_*\eta = m$ such that

$$(Df_x)_*\eta = \eta \quad \text{for } \nu\text{-a.e. } x \in X.$$

In particular, if $\{\eta_y : y \in M\}$ denotes the disintegration of η with respect to m, then

$$Df_x(y)_*\eta_y = \eta_{f_x(y)} \quad \text{for } (\nu \times m)\text{-a.e. } (x, y) \in X \times M.$$

Proof. The argument below follows closely the proof of [13, Theorem 1].

Consider the Markov chain $\{(X_n, Y_n) \colon n \in \mathbb{N}\}$ on $X \times M$ and transition probabilities given by

$$P_{(x,y)}(B \times C) = \nu(B)\delta_{f_x(y)}(C), \quad B \in \mathcal{X}, C \in \mathcal{B}.$$

for all $x \in X, y \in M$. This means that $\{X_n\}$ is a sequence of i.i.d. random variables with values in X and distribution ν, whereas $\{Y_n\}$ is a Markov chain with state space M satisfying the recursive relation

$$Y_{n+1} = f_{X_n}(Y_n).$$

It is straightforward to check that $\mu := \nu \times m$ is a stationary probability for $\{(X_n, Y_n)\}$.

Let $A \colon X \times M \to \mathrm{GL}(d, \mathbb{R})$ be the map defined right before Assumption B: for every $x \in X$ and every $y \in N_x$, let $A(x, y)$ be the matrix representation of $Df_x(y)$ with respect to some global measurable trivialization of TM.

Under the hypotheses of the theorem, we may apply Theorem 2.9 to the Markov chain $\{(X_n, Y_n)\}$ and the matrix function A. This yields a measurable family of probability measures on $\mathbb{P}^{d-1}$,

$$\{\bar{\eta}_{(x,y)} : (x, y) \in X \times M\},$$

and a measurable set $U \subset X \times M$ with $(\nu \times m)(U) = 1$ such that for every $(x, y) \in U$,

$$\bar{\eta}_{(x_1,y_1)} = Df_x(y)_* \, \bar{\eta}_{(x,y)} \quad \text{for } P_{(x,y)}\text{-a.e. } (x_1, y_1) \in X \times M.$$

Using the definition of $P_{(x,y)}$, this condition can be rewritten as follows: for every $(x, y) \in U$,

$$\bar{\eta}_{(x_1,f_x(y))} = Df_x(y)_* \, \bar{\eta}_{(x,y)} \quad \text{for } \nu\text{-a.e. } x_1 \in X. \tag{2.1}$$

It follows that for each $(x, y) \in U$, the measure $\bar{\eta}_{(x_1,f_x(y))}$ is independent of x_1 for ν-almost every $x_1 \in X$.

We now introduce another measurable family $\{\eta_y \colon y \in M\}$ of probability measures on $\mathbb{P}^{d-1}$ defined by

$$\eta_y(B) = \int_X \bar{\eta}_{(x,y)}(B) \, d\nu(x)$$

for every $y \in M$ and every measurable set $B \subset \mathbb{P}^{d-1}$. Then Property (2.1) implies that $Df_x(y)_* \bar{\eta}_{(x,y)} = \eta_{f_x(y)}$ for every $(x, y) \in U$.

Let

$$V = \left\{ (x, y) \in X \times M : \bar{\eta}_{(x,y)} = \eta_y \right\}.$$

This set is measurable, since $(x, y) \mapsto \bar{\eta}_{(x,y)}$ and $(x, y) \mapsto \eta_y$ are measurable maps.

Next, we show that $(\nu \times m)(V) = 1$. For every $x \in X$, let

$$U_x = \{y \in M : (x, y) \in U\}$$

be the x-section of U. Since $(\nu \times m)(U) = 1$, Fubini's Theorem implies that there exists $\bar{x} \in X$ such $m(U_{\bar{x}}) = 1$. By hypothesis, the map $f_{\bar{x}}\colon N_{\bar{x}} \to M$ is an m-preserving C^1 embedding and $m(N_x) = 1$. Therefore, the set $M_{\bar{x}} := f_{\bar{x}}(U_{\bar{x}} \cap N_{\bar{x}})$ is measurable and $m(M_{\bar{x}}) = 1$. By the definition of $U_{\bar{x}}$ and Property (2.1), if $y \in M_{\bar{x}}$, then $(x, y) \in V$ for ν-a.e. $x \in X$. Equivalently,

$$\nu(V^y) = 1 \quad \text{for every } y \in M_{\bar{x}},$$

where $V^y = \{x \in X : (x, y) \in V\}$ is the y-section of V. Finally, by Fubini's Theorem, we obtain

$$(\nu \times m)(V) = \int_M \nu(V^y) dm(y) \geq \int_{M_{\bar{x}}} \nu(V^y) dm(y) = 1.$$

Let $W = U \cap V$. Then W is measurable and satisfies $(\nu \times m)(W) = 1$. Take $(x, y) \in W$. Since $(x, y) \in U$, we have $Df_x(y)_* \bar{\eta}_{(x,y)} = \eta_{f_x(y)}$. On the other hand, $(x, y) \in V$ implies that $\bar{\eta}_{(x,y)} = \eta_y$. Combining these identities, we obtain $Df_x(y)_* \eta_y = \eta_{f_x(y)}$.

To complete the proof, consider the probability measure η on PM whose disintegration over m is given by $\{\eta_y : y \in M\}$. By the conclusion above, η satisfies

$$(Df_x)_* \eta = \eta \quad \text{for } \nu\text{-a.e. } x \in X.$$

□

The following corollary is an immediate consequence of Theorem 2.10 applied to random additive perturbations of toral maps, together with the triviality of the projective bundle over $\mathbb{T}^d$, i.e. $P\mathbb{T}^d \cong \mathbb{T}^d \times \mathbb{P}^{d-1}$.

Corollary 2.11. *Let F be ν-random additive perturbation of a toral map g. Suppose that*

$$\lambda_F^+(\omega, y) = 0 \quad \text{for } \rho_\nu \times m\text{-a.e. } (\omega, y) \in \Omega \times \mathbb{T}^d.$$

Then there exists a measurable family of probability measures $\{\eta_y : y \in \mathbb{T}^d\}$ on $\mathbb{P}^{d-1}$ such that for m-a.e. $y \in M$,

$$Dg(y)_* \eta_y = \eta_{g(y)+x} \quad \text{for } \nu\text{-a.e. } x \in \mathbb{T}^d.$$

2.3.3. *When ν is absolutely continuous.* We now examine some implications of Corollary 2.11 when ν is absolutely continuous with respect to vol.

For each $y \in \mathbb{T}^d$, let $B_\epsilon(y)$ denote the closed ball in $\mathbb{T}^d$ with radius $\epsilon > 0$ centered at y. Define $\mathrm{vol}_{B_\epsilon(y)}$ to be the normalized restriction of the Riemannian volume vol to $B_\epsilon(y)$.

Let $\mu = (Dg)_* m$ be the probability measure on $\mathrm{SL}(d, \mathbb{R})$ obtained as the push-forward of m under the map $Dg\colon N \to \mathrm{SL}(d, \mathbb{R})$. Denote by $H \subset \mathrm{SL}(d, \mathbb{R})$ the support of μ. For $h \in H$, we use the same symbol h to denote both the matrix and its induced action on the projective space $\mathbb{P}^{d-1}$.

Proposition 2.12. *Let F be a ν-random additive perturbation of a toral map g with $\nu \ll \mathrm{vol}$ and $B_\epsilon(0) \subseteq \operatorname{supp}\nu$ for some $\epsilon > 0$. Suppose that*

$$\lambda_F^+(\omega, y) = 0 \quad \text{for } (\rho_\nu \times m)\text{-a.e. } (\omega, y) \in \Omega \times \mathbb{T}^d.$$

Then there exists a probability measure η on $\mathbb{P}^{d-1}$ such that

$$h_*\eta = \eta \quad \text{for all } h \in H. \tag{2.2}$$

Moreover, one of the following must hold:

1. *H is contained in a compact subgroup of $\mathrm{SL}(d, \mathbb{R})$, or*
2. *there exists a nonempty set $L \subset \mathbb{R}^d$ consisting of finitely many proper subspaces such that $h(L) = L$ for every $h \in H$.*

Proof. By Corollary 2.11, there exists a measurable family $\{\eta_y : y \in \mathbb{T}^d\}$ of probability measures on $\mathbb{P}^{d-1}$ and a measurable set $U \subset N$ with $m(U) = 1$ such that for every $y \in U$,

$$Dg(y)_*\eta_y = \eta_{g(y)+x} \quad \text{for } \mathrm{vol}\text{-a.e. } x \in B_\epsilon(0).$$

Let $V = g(U \cap N)$. Since $g|_N$ is a C^1 embedding and $m(g(N)) = 1$, the set V is measurable and satisfies $m(V) = 1$. Moreover, since m is the normalized Riemannian volume on $\mathbb{T}^d$, it follows that V is dense in $\mathbb{T}^d$.

By the hypotheses on the measure ν, for each $z \in V$,

$$w \mapsto \eta_w \text{ is constant } \mathrm{vol}\text{-a.e. on } B_\epsilon(z).$$

As a consequence, if $z_1, z_2 \in V$ and the interiors of the balls $B_\epsilon(z_1)$ and $B_\epsilon(z_2)$ intersect, then $w \mapsto \eta_w$ is constant vol-a.e. on $B_\epsilon(z_1) \cup B_\epsilon(z_2)$.

Fix $z_1, z_2 \in V$. Since $\mathbb{T}^d$ is connected and V is dense in $\mathbb{T}^d$, there exists an ϵ-chain $w_1 = z_1, w_2, \ldots, w_N = z_2$ contained in V connecting z_1 to z_2. Hence, for each $i = 0, \ldots, N-1$, we have

$$\mathrm{vol}(B_\epsilon(w_i) \cap B_\epsilon(w_{i+1})) > 0.$$

By the previous observation, this implies $\eta_{z_1} = \eta_{z_2}$. Since z_1 and z_2 were arbitrary, it follows that $z \mapsto \eta_z$ is constant on V. Therefore, there exists a probability measure η on $\mathbb{P}^{d-1}$ such that $\eta_z = \eta$ for every $z \in V$, and

$$Dg(y)_*\eta = \eta \quad \text{for all } y \in U \cap V. \tag{2.3}$$

We now extend this identity to all $y \in N$. Fix $y \in N$. Since $m(U \cap V) = 1$, there exists a sequence $\{y_n\} \subset U \cap V$ such that $y_n \to y$. By the continuity of Dg at y and identity (2.3), we obtain

$$Dg(y)_*\eta = \lim_{n\to+\infty} Dg(y_n)_*\eta = \lim_{n\to+\infty} \eta = \eta. \tag{2.4}$$

Next, observe that $\operatorname{supp} \mu \subset \overline{Dg(N)}$, since g is differentiable at least on N. Therefore, if $h \in \operatorname{supp} \mu$, then there exists a sequence $\{y_n\} \subset N$ such that $Dg(y_n) \to h$. Since $\mathbb{T}^d$ is compact, by passing to a subsequence, we may assume without loss of generality that $y_n \to y$ for some $y \in \mathbb{T}^d$. Then, by (2.4),

$$h_*\eta = \lim_{n\to+\infty} Dg(y_n)_*\eta = \lim_{n\to+\infty} \eta = \eta.$$

The final part of the theorem follows from (2.2) and standard results on linear cycles [33, Sections 6.3 and 7.3]. □

Remark 2.13. When $d = 2$, the last part of Proposition 2.12 reduces to the following characterization (see [33, Section 6.3]):

(1) H is contained in a compact subgroup of $\mathrm{SL}(2, \mathbb{R})$, or
(2) there exists a nonempty set $L \subset \mathbb{R}^2$ consisting of one or two subspaces such that $h(L) = L$ for every $h \in H$.

The second case can be further refined into two complementary possibilities:

- L consists of either one or two invariant subspaces,
- L consists of two subspaces interchanged by some $h \in H$.

Remark 2.14. The failure of Conditions (1) and (2) in the conclusion of Proposition 2.12 implies that $\lambda_F^+ > 0$ a.e. on $\Omega \times \mathbb{T}^d$. This provides a criterion for the positivity of the maximal Lyapunov exponent of F analogous to Furstenberg's criterion for random matrices [33, Section 7.3].

The following corollary provides a simple condition ensuring that $\lambda_F^+ > 0$ a.e. on $\Omega \times \mathbb{T}^d$ when $d = 2$. It will be used in Section 4 to establish the positivity of λ_F^+ when F is a random additive perturbation of a standard map.

Corollary 2.15. *Assume that the hypotheses of Proposition 2.12 hold and that $d = 2$. Suppose there exist elements $h_1, h_2 \in H$ such that $h_1 \neq \pm I$, $|\operatorname{tr} h_1| \geq 2$ and $0 < |\operatorname{tr} h_2| < 2$. Then $\lambda_F^+(\omega, y) > 0$ for $(\rho_v \times m)$-a.e. $(\omega, y) \in \Omega \times \mathbb{T}^2$.*

Proof. The conditions $h_1 \neq \pm I$ and $|\operatorname{tr} h_1| \geq 2$ imply that h_1 is hyperbolic or parabolic but not diagonalizable, while $0 < |\operatorname{tr} h_2| < 2$ ensures that h_2 is elliptic but not conjugate to a rotation by angle $\pi/2$. These properties contradicts Conditions (1) and (2) of Proposition 2.12, respectively, and therefore imply that $\lambda_F^+ > 0$ a.e. on $\Omega \times \mathbb{T}^d$. □

The next corollary follows directly from Proposition 2.12.

Corollary 2.16. *Assume that the hypotheses of Proposition 2.12 hold. If there exists a sequence $\{h_n\}_{n\geq 0}$ contained in H and a probability measure ζ on $\mathbb{P}^{d-1}$ such that $(h_n)_*\eta \longrightarrow \zeta$ as $n \to +\infty$ in the weak-* topology, then $\zeta = \eta$.*

The following lemma addresses a special case of Corollary 2.16 in which $\|h_n\| \to +\infty$ as $n \to +\infty$, a situation that arises in the study of billiards in Section 3.

Corollary 2.17. *Assume that the hypotheses of Proposition 2.12 hold and that $d = 2$. Suppose there exists a sequence $\{h_n\}_{n\geq 0} \subset H$ such that*

$$h_n = A_n D_n B_n,$$

where

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix}, \qquad \alpha_n \xrightarrow[n\to\infty]{} 0,$$

$$D_n = \begin{pmatrix} 1+a_n & b+b_n \\ c_n & 1+d_n \end{pmatrix}, \quad b \neq 0, \quad a_n, b_n, c_n, d_n \xrightarrow[n\to\infty]{} 0,$$

$$B_n = \beta_n \begin{pmatrix} \beta_n^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \beta_n \xrightarrow[n\to\infty]{} +\infty.$$

Then

$$(h_n)_* \eta \xrightarrow[n\to\infty]{\text{weak-}*} \delta_{\widehat{e}},$$

and consequently

$$\eta = \delta_{\widehat{e}},$$

where $\widehat{e} \in \mathbb{P}^1$ denotes the point with homogeneous coordinates $[1:0]$.

Proof. By hypothesis, we have

$$h_n = \beta_n \begin{pmatrix} \beta_n^{-1}(1+a_n) & b+b_n \\ \beta_n^{-1}\alpha_n c_n & \alpha_n(1+d_n) \end{pmatrix}.$$

Let $\widehat{h}_n \colon \mathbb{P}^1 \to \mathbb{P}^1$ denote the map induced by the action of h_n on $\mathbb{P}^1$, and let $\widehat{e} \in \mathbb{P}^1$ be the point with homogeneous coordinates $[1:0]$. For any $[u:v] \in \mathbb{P}^1$, we have

$$\begin{aligned} \widehat{h}_n([u:v]) &= \left[\beta_n^{-1}(1+a_n)u + (b+b_n)v : \beta_n^{-1}\alpha_n c_n u + \alpha_n(1+d_n)v\right] \\ &\xrightarrow[n\to\infty]{} \begin{cases} [bv:0] = \widehat{e}, & \text{if } v \neq 0, \\ [u:0] = \widehat{e}, & \text{if } v = 0 \text{ and } u \neq 0. \end{cases} \end{aligned}$$

Thus, the sequence of maps $\widehat{h}_n$ converges pointwise to the constant map

$$L \colon \mathbb{P}^1 \to \mathbb{P}^1, \qquad L([u:v]) = \widehat{e} \quad \text{for all } [u:v] \in \mathbb{P}^1.$$

By the Dominated Convergence Theorem, it follows that

$$\widehat{h}_{n*}\eta \xrightarrow[n\to\infty]{\text{w}^*} L_*\eta = \delta_{\widehat{e}}.$$

Therefore, by Corollary 2.16, we conclude that $\eta = \delta_{\widehat{e}}$. □

2.4. **Some ergodic properties of random perturbations of continuous toral maps.** The setting of this section coincides with that of Section 2.2: we consider a random additive perturbation F of a map g on $\mathbb{T}^d$. However, we impose additional assumptions on this system. Specifically, we assume that

(1) $g : \mathbb{T}^d \to \mathbb{T}^d$ is a continuous map defined on all of $\mathbb{T}^d$,
(2) the probability measure ν is absolutely continuous with respect to vol.

This framework is relevant for the applications considered here, since both the billiard map associated with a convex table and the standard map are continuous and volume preserving. Consequently, the random perturbations studied in Sections 3.3 and 4 fall within this setting.

Denote by $\{P_y : y \in \mathbb{T}^d\}$ the transition probabilities of the Markov chain $\{Y_n\}$ with state space $\mathbb{T}^d$ defined recursively by

$$Y_{n+1} = f_{X_n}(Y_n) = g(Y_n) + X_n, \tag{2.5}$$

where $\{X_n\}$ is a sequence of i.i.d. random variables with distribution ν. It is straightforward to check that for every measurable subset $E \subset \mathbb{T}^d$,

$$P_y(E) = \nu\big(\{x \in \mathbb{T}^d : g(y) + x \in E\}\big).$$

We show that the Markov chain on $\mathbb{T}^d$ generated by the random maps f_{X_n} is uniquely ergodic whenever ν is absolutely continuous with respect to vol.

As a consequence, the measure m is ergodic for the Markov chain $\{Y_n\}$, and the measure $\rho_\nu \times m$ is ergodic for the skew-product F. In particular, this applies to random perturbations of billiard maps studied in Section 3.4.

Recall that the Markov kernel $\{P_y : y \in \mathbb{T}^d\}$ is called *strong Feller* if the associated Markov operator $P\varphi(y) := \int \varphi(z)\, dP_y(z)$ maps every bounded measurable function φ to a continuous function on $\mathbb{T}^d$. In the following, we sometimes write dz instead of $d\,\mathrm{vol}(z)$.

Lemma 2.18. *If* $\nu \ll \mathrm{vol}$, *then the Markov kernel* $\{P_y : y \in \mathbb{T}^d\}$ *is strong Feller.*

Proof. Suppose that $d\nu(z) = p(z)dz$ for some non-negative $p \in L^1(dz)$. For $w \in \mathbb{T}^d$, let $U_w \colon L^1(dz) \to L^1(dz)$ denote the translation operator defined by $U_w\varphi(z) = \varphi(z - w)$ for every $\varphi \in L^1(dz)$. It is well-known that $\{U_w : w \in \mathbb{T}^d\}$ forms a strongly continuous group on $L^1(dz)$ [22].

Let $\varphi\colon \mathbb{T}^d \to \mathbb{R}$ be bounded and measurable. For any $y \in \mathbb{T}^d$, using the translation-invariance of dz, we obtain

$$\begin{aligned} P\varphi(y) &= \int \varphi(z)\, dP_y(z) = \int \varphi(g(y)+z) d\nu(z) = \int \varphi(g(y)+z)p(z)dz \\ &= \int \varphi(z)p(z-g(y))dz = \int \varphi(z)U_{g(y)}p(z)dz. \end{aligned}$$

Let $y_1, y_2 \in \mathbb{T}^d$. Since g is continuous and $\{U_w\}$ is strongly continuous, we have

$$\begin{aligned} |P\varphi(y_1) - P\varphi(y_2)| &= \left| \int \varphi(z) \left(U_{g(y_1)}p(z) - U_{g(y_2)}p(z) \right) dz \right| \\ &\leq \|\varphi\|_\infty \left\| U_{g(y_1)}p - U_{g(y_2)}p \right\|_1 \xrightarrow[y_2 \to y_1]{} 0. \end{aligned}$$

Hence, $P\varphi$ is continuous, proving that $\{P_y : y \in \mathbb{T}^d\}$ is strong Feller. □

Proposition 2.19. *The probability measure m is the only stationary measure of $\{P_y : y \in \mathbb{T}^d\}$, and $\rho_\nu \times m$ is an ergodic invariant probability of F.*

Proof. Since $\{P_y\}$ is strong Feller by Lemma 2.18 and the support of m is the whole $\mathbb{T}^d$, it follows from [4, Part (iii) of Proposition 5.18] that $\{P_y\}$ is uniquely ergodic. Consequently, m is an ergodic stationary probability measure for $\{P_y\}$. Finally, by [33, Proposition 5.13], the invariant probability measure $\rho_\nu \times m$ of F is ergodic. □

Remark 2.20. Since the probability measure $\rho_\nu \times m$ is an ergodic by Proposition 2.19, it follows that $\lambda_F^-(\omega, y)$ and $\lambda_F^+(\omega, y)$ are constant $(\rho_\nu \times m)$-a.e. on $\Omega \times \mathbb{T}^d$.

The properties of the Markov kernel $\{P_y : y \in \mathbb{T}^d\}$ allow us to establish the following equidistribution result for random orbits on $\mathbb{T}^d$.

Proposition 2.21. *Let $y \in \mathbb{T}^d$. Then, for ρ_ν-almost every $\omega \in \Omega$, the random orbit $\{F_\omega^n(y)\}_{n \geq 0}$ is equidistributed with respect to m, that is,*

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{F_\omega^k(y)} \xrightarrow[n \to \infty]{\mathrm{w}^*} m \quad \text{for } \rho_\nu\text{-a.e. } \omega \in \Omega.$$

In particular, ρ_ν-almost every random orbit of y is dense in $\mathbb{T}^d$.

Proof. By Breiman's Law of Large Numbers for Markov chains [5, Corollary 2.7], applied to the uniquely ergodic kernel P_y, the equidistribution property holds; see also [23, Lemma 7.1].

The density of the random orbit is an immediate consequence of equidistribution and the fact that the measure m assigns positive measure to every nonempty open subset of $\mathbb{T}^d$. □

3. VANISHING LYAPUNOV EXPONENTS CHARACTERIZE CIRCULAR BILLIARDS

3.1. **Convex billiards on surfaces with constant curvature.** In this section, we recall the fundamental definitions and properties of billiards in convex domains on surfaces of constant curvature. For a comprehensive treatment of billiards on general surfaces, we refer to see [17], and for the case of surfaces with constant curvature to [19].

Let S denote one of the standard surfaces of constant curvature K: the Euclidean plane $\mathbb{E}^2$ ($K = 0$), the sphere $\mathbb{S}^2$ ($K = 1$) and the hyperbolic plane $\mathbb{H}^2$ ($K = -1$). Let $D \subset S$ be a domain whose boundary ∂D is a C^2 simple closed convex curve with positive geodesic curvature (such a curve is called an *oval* in [19, 17]). We parametrize ∂D by arc-length s, normalized so that the total length of the curve is 1, i.e. $|\partial D| = 1$.

A *billiard* with table D is the mechanical system consisting of a point particle that moves inside D along geodesics of S and reflects elastically upon colliding with the boundary ∂D, obeying to the usual law of reflection: the angles of reflection equals to angle of incidence. We refer to such systems as *convex billiards*.

3.1.1. *Billiard map in coordinates* (s, θ). Each collision of the particle with ∂D is described by the pair (s, θ), where s denotes the arc-length parameter (mod 1) of the point of impact, and $\theta \in [0, \pi]$ is the angle between the incoming trajectory and the positively oriented tangent to ∂D at s. Hence, the space of all possible collisions is the closed cylinder

$$Q = S^1 \times [0, \pi]$$

The billiard map associated with the table D on a surface S is the map $\phi\colon Q \to Q$ that assigns to each collision $(s, \theta) \in Q$ the next collision

$$\phi(s, \theta) = (s_1(s, \theta), \theta_1(s, \theta)).$$

The map ϕ for general surfaces of constant curvature retains the same properties of the billiard map for planar convex billiards.

Proposition 3.1 ([17]). *The map $\phi\colon Q \to Q$ satisfies the following properties:*

1. *ϕ is a homeomorphism,*
2. $\operatorname{int} Q := S^1 \times (0, \pi)$ *and* $\partial Q := S^1 \times \{0, \pi\}$ *are ϕ-invariant sets,*
3. $\phi|_{\operatorname{int} Q}\colon \operatorname{int} Q \to \operatorname{int} Q$ *is a twist C^1 diffeomorphism,*
4. $\phi|_{\partial Q} = \mathrm{id}_{\partial Q}$,
5. *ϕ preserves the measure* $d\bar{m} = \sin\theta\, ds\, d\theta$.

Let $\kappa(s)$ denote the geodesic curvature of ∂D at the point with arc-length parameter s. Let $t(s, \theta)$ be the geodesic distance on S between the points of ∂D corresponding to s and $s_1(s,\theta)$. Write

$$t = t(s,\theta), \quad \theta_1 = \theta_1(s,\theta), \quad \kappa = \kappa(s), \quad \kappa_1 = \kappa(s_1(s,\theta)).$$

Then for every $(s,\theta) \in \operatorname{int} Q$, the derivative $D\phi(s,\theta)$ is given by following expressions, corresponding respectively to the cases $S = \mathbb{E}^2$, $S = \mathbb{S}^2$ and $S = \mathbb{H}^2$ [15, 27, 19]:

$$D\phi(s,\theta) = \begin{pmatrix} \frac{\kappa t - \sin\theta}{\sin\theta_1} & \frac{t}{\sin\theta_1} \\ \frac{\kappa_1 \kappa t - \kappa_1 \sin\theta - \kappa \sin\theta_1}{\sin\theta_1} & \frac{\kappa_1 t - \sin\theta_1}{\sin\theta_1} \end{pmatrix},$$

$$D\phi(s,\theta) = \begin{pmatrix} \frac{\kappa \sin t - \cos t \sin\theta}{\sin\theta_1} & \frac{\sin t}{\sin\theta_1} \\ \frac{\sin t(\kappa\kappa_1 - \sin\theta\sin\theta_1) - \cos t(\kappa_1 \sin\theta + \kappa \sin\theta_1)}{\sin\theta_1} & \frac{\kappa_1 \sin t - \cos t \sin\theta_1}{\sin\theta_1} \end{pmatrix},$$

$$D\phi(s,\theta) = \begin{pmatrix} \frac{\kappa \sinh t - \cosh t \sin\theta_1}{\sin\theta_1} & \frac{\sinh t}{\sin\theta_1} \\ \frac{\sinh t(\kappa\kappa_1 + \sin\theta\sin\theta_1) - \cosh t(\kappa_1 \sin\theta + \kappa \sin\theta_1)}{\sin\theta_1} & \frac{\kappa_1 \sinh t - \cosh t \sin\theta_1}{\sin\theta_1} \end{pmatrix}. \tag{3.1}$$

Moreover, in all cases, the derivative $D\phi$ admits the following limits [27] and [17, Proposition 3.4]: for every $s \in S^1$,

$$\lim_{\theta \to 0^+} D\phi(s,\theta) = \lim_{\theta \to \pi^-} D\phi(s,\theta) = \begin{pmatrix} 1 & \frac{2}{\kappa(s)} \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

3.1.2. *Billiard map in coordinates* (s,r). For the purposes of this work, it is convenient to replace the coordinate θ by $r = -\cos\theta \in [-1,1]$. In coordinates (s,r), the set of all collisions is given by the closed cylinder $V = S^1 \times [-1,1]$, and the billiard map is denoted by $\Phi\colon V \to V$.

The map Φ inherits the properties of ϕ described in Proposition 3.1. Namely: 1) Φ is a homeomorphism, 2) the sets $\operatorname{int} V := S^1 \times (-1,1)$ and $\partial V := S^1 \times \{-1,1\}$ are invariant under Φ, 3) $\Phi|_{\operatorname{int} V}$ is a twist C^1 diffeomorphism, 4) $\Phi_{\partial V} = \mathrm{id}_{\partial V}$, 5) Φ preserves the measure $dsdr$.

However, a key difference between the two maps is that $\det D\Phi \equiv 1$ on $\operatorname{int} V$, whereas for Φ, one has $\det \phi = \sin\theta / \sin\theta_1 \not\equiv 1$ on $\operatorname{int} Q$. This fact is the reason for working with the map Φ rather than with ϕ.

3.1.3. *Billiard map on the* 2*-torus.* We now introduce a map T on the 2-torus, induced by the billiard map Φ. Random perturbations of T will be studied in Section 3.3.

Since the restriction of Φ to ∂V is the identity, we can define an automorphism T of the torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\mathbb{Z}$ endowed with the flat metric $ds^2 + dr^2$ by choosing $R := [0,1) \times [-1,1)$ as a fundamental domain of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\mathbb{Z}$ and setting

$$T(s,r) = \Phi(s,r) \quad \text{for all } (s,r) \in R.$$

From the properties of Φ, one can immediately deduce the following properties of T: 1) $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism, 2) the sets $\operatorname{int} R := [0,1) \times (-1,1)$ and $\partial R := [0,1) \times \{-1\}$ are invariant under T, 3) the restriction of T to $\operatorname{int} R$ is a C^1 diffeomorphism and $\det DT(s,r) = 1$ for all $(s,r) \in \operatorname{int} R$, 4) $T(s,-1) = (s,-1)$ for every $s \in [0,1)$, 5) T preserves the Riemannian volume measure vol on $\mathbb{T}^2$ given by $d\,\mathrm{vol} = dsdr$.

3.2. **Circular billiards.** Let $D \subset S$ be a geodesic disk. Since S has constant curvature, the boundary ∂D is necessarily a geodesic circle. In this paper, we restrict our attention to geodesic disks whose boundaries ∂D have positive geodesic curvature.

Under these assumptions, the billiard map $\phi\colon Q \to Q$ in coordinates (s,θ) associated with the geodesic disk takes the simple form

$$\phi(s,\theta) = (s + \alpha(s), \theta) \quad \text{for all } (s,\theta) \in Q,$$

for a suitable smooth function α on $\operatorname{int} Q$; see [11, 20]. Similarly, the corresponding map $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ has the form

$$T(s,\theta) = (s + \beta(r), r) \quad \text{for all } (s,\theta) \in \mathbb{T}^2,$$

where $\beta(r) = \alpha(-\arccos r)$. Both maps ϕ and T are integrable.

3.3. **Random additive perturbations of billiards.** Let $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the billiard map associated with a convex domain D on a surface of constant curvature. We now consider random additive perturbations of T. In the framework of Section 2, these are described by the skew-product

$$F(\omega, y) = (\sigma(\omega), f_{\omega_0}(y)) \quad \text{for every } (\omega, y) \in \Omega \times M,$$

where $\Omega = (\mathbb{T}^2)^{\mathbb{N}}$ is endowed with the probability measure $\rho_\nu = \nu^{\mathbb{N}}$ for some probability measure ν on $\mathbb{T}^2$, and $M = \mathbb{T}^2$ is equipped with the flat Riemannian metric. The additive perturbation f_x, $x \in \mathbb{T}^2$ of T is defined by

$$f_x(y) = T(y) + x \quad \text{for all } y = (s,r) \in \mathbb{T}^2.$$

Note that a random additive perturbation F of T is completely determined by the choice of the probability measure ν.

Lemma 3.2. *For any choice of the probability measure ν on $\mathbb{T}^2$, the map $f(x,y) = f_x(y)$ and ν satisfy Assumptions A and B with $N = \operatorname{int} R$ and the Borel probability measure m being the normalized Riemannian volume on $\mathbb{T}^2$.*

Proof. Assumption A follows directly from the properties of T. In particular, we have $\mathrm{vol}(M \setminus N) = 0$.

To verify Assumption B, we need to show that the functions $(x,y) \mapsto \log^+ \|Df_x(y)\|$ and $(x,y) \mapsto \log^+ \|(Df_x(y))^{-1}\|$ belong to $L^1(\nu \times m)$ for any choice of ν. Since $Df_x = DT$ for every $x \in M$, $DT = D\Phi$ on $\operatorname{int} R$ and $m(\partial R) = 0$, it is enough to show that $\log^+ \|D\Phi\|$

and $\log^+ \|D\Phi^{-1}\|$ are integrable with respect to the measure $dsdr$ on $[0,1] \times [-1,1]$. We present the proof only for the integrability $\log^+ \|D\Phi\|$; the argument for $\log^+ \|D\Phi^{-1}\|$ is similar.

Let $h\colon [0,1] \times [-\pi,\pi] \to [0,1] \times [-1,1]$ be the change of coordinates given by $h(s,\theta) = (s, -\cos\theta)$. Hence $\Phi = h \circ \phi \circ h^{-1}$. Since $\|Dh(s,\theta)\| = 1$ for all $(s,\theta) \in [0,1] \times [-\pi,\pi]$, and $\|Dh^{-1}(s,r)\| = 1/\sqrt{1-r^2}$ for all $(s,r) \in [0,1] \times (-1,1)$, it follows that

$$\log \|D\Phi(s,r)\| \le \log \|D\phi(h^{-1}(s,r))\| - \frac{1}{2}\log(1-r^2).$$

It is straightforward to verify that the function $\log(1-r^2)$ is integrable with respect to the measure $ds\,dr$. Since the map $(s,\theta) \mapsto t(s,\theta)$ is bounded, it follows from the expression of $D\phi$ that there exists a constant $C > 0$ such that, for all $(s,r) \in [0,1] \times (-1,1)$, we have

$$\|D\phi(h^{-1}(s,r))\| \le \frac{C}{\sin\theta_1(h^{-1}(s,r))} = \frac{C}{\sqrt{1-(r_1(r,s))^2}},$$

and thus,

$$\log \|D\phi(h^{-1}(s,r))\| \le \log C - \frac{1}{2}\log(1-(r_1(r,s))^2).$$

By the invariance of the measure $ds\,dr$, we have

$$\int_{[0,1]\times[-1,1]} \log(1-(r_1(r,s))^2)\,ds\,dr = \int_{[0,1]\times[-1,1]} \log(1-r^2)\,ds\,dr.$$

Therefore $\log(1-(r_1(r,s))^2)$ is also integrable with respect to $ds\,dr$. This completes the proof. □

For notational convenience, we denote the extremal Lyapunov exponents of F by λ^- and λ^+, omitting the subscript F.

We now show that when the billiard table D is a geodesic disk, the Lyapunov exponents $\lambda^-(\omega,y)$ and $\lambda^+(\omega,y)$ vanish for any choice of the probability measure ν.

Lemma 3.3. *Suppose that D is a geodesic disk. Then for every probability measure ν on $\mathbb{T}^2$, we have*

$$\lambda^-(\omega,y) = \lambda^+(\omega,y) = 0 \quad \textit{for } \rho_\nu \times m\textit{-a.e. } (\omega,y) \in \Omega \times \mathbb{T}^2.$$

Proof. Let ν be a probability measure on $\mathbb{T}^2$. In the following, we write a.e. for $\rho_\nu \times m$-a.e. on $\Omega \times \mathbb{T}^2$.

Let T be the billiard map associated with a geodesic disk D. Its explicit expression is given in Section 3.2. For all $x \in \mathbb{T}^2$, we have

$$Df_x(s,r) = DT(s,r) = \begin{pmatrix} 1 & \beta'(r) \\ 0 & 1 \end{pmatrix} \quad \text{for all } (s,r) \in \operatorname{int} R.$$

It follows that

$$Df_x(y)\,\frac{\partial}{\partial s} = \frac{\partial}{\partial s} \quad \text{for all } (x,y) \in \mathbb{T}^2 \times \operatorname{int} R.$$

By Oseledets' Theorem, at least one of the Lyapunov exponents λ^- or λ^+ must vanish a.e. Since $\lambda^- + \lambda^+ = 0$ almost everywhere, we conclude that both Lyapunov exponents vanish a.e., as claimed. □

3.4. **Vanishing Lyapunov exponents and integrability.** We now assume that $\nu \ll \mathrm{vol}$ and that $B_\epsilon(0) \subseteq \operatorname{supp} \nu$ for some $\epsilon > 0$, and proceed to prove Theorem 1.1.

For such a ν, all the conclusions of Section 2.3.3 apply to the skew-product F. In particular, the probability measure $\rho_\nu \times m$ is ergodic, and the Lyapunov exponents λ^- and λ^+ are constant for $\rho_\nu \times m$-almost every $(\omega, y) \in \Omega \times \mathbb{T}^2$. We emphasize, however, that this property will not be used in the proofs of the propositions that follow.

Recall that $\phi\colon Q \to Q$ is the billiard map in coordinates $(s,\theta) \in Q$ with $Q = S^1 \times [0,\pi]$.

Proposition 3.4. *Suppose $\lambda^+(\omega, y) = 0$ for $\rho_\nu \times m$-a.e. $(\omega, y) \in \Omega \times \mathbb{T}^2$. Then there exists a strictly positive continuous function $\gamma\colon \operatorname{int} Q \to \mathbb{R}$ such that*

$$D\phi(s,\theta)\frac{\partial}{\partial s} = \gamma(s,\theta)\frac{\partial}{\partial s} \quad \textit{for all } (s,\theta) \in \operatorname{int} Q.$$

Proof. By Lemma 3.2, Proposition 2.12 applies to F. Moreover, the expression of $D\phi$ in (3.1) and the property (3.2), together with the definitions of Φ and T, show that DT admits a factorization required by Corollary 2.17, which therefore applies to F. Since DT is continuous on $[0,1) \times (-1,1)$, it follows that $\{DT(s,r)\colon (s,r) \in [0,1) \times (-1,1)\}$ is contained in the support of $DT_* m$. Hence, by Proposition 2.12 and Corollary 2.17, we obtain

$$DT(s,r)_*\delta_{\widehat{e}} = \delta_{\widehat{e}} \quad \text{for all } (s,r) \in [0,1) \times (-1,1).$$

where $\widehat{e}$ is the element of $\mathbb{P}^1$ with homogeneous coordinates $[1:0]$.

This implies that $\widehat{e}$ is a fixed point of the projective action of $DT(s,r)$ for all $(s,r) \in [0,1) \times (-1,1)$. Equivalently, the subspace L_s spanned by the vector $\partial/\partial s$ is invariant with respect to $DT(s,r)$ for every $(s,r) \in [0,1) \times (-1,1)$.

In view of the definition of T, we see that L_s is also invariant under $D\phi(s,\theta)$ for every $(s,\theta) \in \operatorname{int} Q$. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{R}^2$. Thus, we have

$$D\phi(s,\theta)\frac{\partial}{\partial s} = \gamma(s,\theta)\frac{\partial}{\partial s} \quad \text{for every } (s,\theta) \in \operatorname{int} Q,$$

where $\gamma(s,\theta) := \langle \partial/\partial s, D\phi(s,\theta)\partial/\partial s \rangle$. Since ϕ is a C^1 diffeomorphism on $\operatorname{int} Q$, the function γ is well-defined and continuous on

$\operatorname{int} Q$. Moreover, γ must be either strictly positive or strictly negative, because $\gamma(s,\theta) = 0$ for some $(s,\theta) \in \operatorname{int} Q$ would imply $D\phi(s,\theta)\partial/\partial s = 0$, contradicting the invertibility of $D\phi(s,\theta)$. To establsh that γ is strictly positive, it suffices to show that for fixed s, we have $\lim_{\theta\to 0^+} \gamma(s,\theta)$ exists and is positive. Indeed, by (3.2), we have $\lim_{\theta\to 0^+} \gamma(s,\theta) = \lim_{\theta\to 0^+} \langle D\phi(s,\theta)\partial/\partial s, \partial/\partial s\rangle = 1$. □

Theorem 3.5. *$\lambda^+(\omega,y) = 0$ for $\rho_\nu \times m$-a.e. $(\omega,y) \in \Omega \times \mathbb{T}^2$ if and only if D is a geodesic disk.*

Proof. If D is a geodesic disk, then $\lambda^+ = 0$ for $\rho_\nu \times m$-a.e. $(\omega,y) \in \Omega \times \mathbb{T}^2$ by Lemma 3.3.

Conversely, assume that $\lambda^+ = 0$ for $\rho_\nu \times m$-a.e. $(\omega,y) \in \Omega \times \mathbb{T}^2$. In the remainder of the proof, we first work with the billiard map Φ in coordinates (s,r), and later return to the map ϕ.

Let E be the sub-bundle of TV defined by

$$E(s,r) = \operatorname{span}\left(\frac{\partial}{\partial s}\right) \quad \text{for every } (s,r) \in V.$$

By Proposition 3.4, we have

$$D\Phi(s,r)E(s,r) = E(\Phi(s,r)) \quad \text{for all } (s,r) \in \operatorname{int} V.$$

Note that the integral curves of the sub-bundle E are the circles Γ_r, corresponding to the level sets of the function $p(s,r) := r$. Since E is $D\Phi$-invariant, the foliation $\{\Gamma_r\}$ is Φ-invariant, i.e. $\Phi(\Gamma_r) = \Gamma_{p(\Phi(s,r))}$ for every $r \in [-1,1]$.

This implies that Φ must be a skew-product of the form $\Phi(s,r) = (b(s,r), a(r))$ for some maps $a\colon [-1,1] \to [-1,1]$ and $b\colon V \to S^1$ such that a is a homeomorphism fixing the points $r = -1$ and $r = 1$ and a C^1 diffeomorphism on $(-1,1)$, whereas b is continuous and C^1 on the interior of V. Since Φ preserves $dsdr$, the measure $dr = p_* dsdr$ must be a-invariant. This together the fact that $r = -1$ and $r = 1$ are fixed point s of a implies that a is the identity. Thus $\Phi(s,r) = (b(s,r), r)$. It follows that each circle Γ_r is Φ-invariant.

The same is true for the map ϕ: it is a skew product of the form $\phi(s,\theta) = (\bar{b}(s,\theta), \theta)$ for some C^1 function $\bar{b}$. Note also that in this case $\det D\phi = \sin\theta / \sin\theta_1 = 1$. As a consequence, the entries on the main diagonal of the matrix of $D\phi$ must be equal to 1. Comparing with the expression of $D\phi$ in (3.1), we conclude that the curvature k of ∂D is constant. Hence, D is a geodesic disk. □

Remark 3.6. The second part of the proof of Theorem 3.5 can alternatively be derived from a result of Bialy [8, 9], which characterizes circular billiard tables among all convex tables on surfaces of constant curvature. Specifically, Bialy proved that circular billiards are the only ones for which every trajectory has no conjugate points. Moreover, he showed that the absence of conjugate points is equivalent to the existence of a measurable monotone invariant sub-bundle. By Proposition 3.4, the sub-bundle E introduced in the proof of Theorem 3.5 has this property. Thus, Bialy's result applies, and we may conclude that D is a geodesic disk. However, our approach is more direct, since it relies on the specific form of the invariant sub-bundle E derived in Proposition 3.4. For an alternative proof of Bialy's theorem in the planar case, see also [36].

Remark 3.7. We discuss a possible extension of Theorem 3.5 to magnetic billiards. In these systems, a charged point particle moves under the influence of a constant magnetic field. Unlike standard billiards, where the particle travels along geodesics between elastic reflections at the boundary, the motion in magnetic billiards is governed by the Lorentz force, so that trajectories are curves of constant geodesic curvature determined by the field strength; for instance, see [6, 26]. Upon collision with the boundary, the particle undergoes specular reflection.

For convex planar tables, Berglund and Kunz [6] showed that the magnetic billiard map shares many qualitative features of the corresponding non-magnetic billiard map, with one notable exception: among the two boundary components of the phase space, only the circle $S^1 \times \{-1\}$ is invariant under the magnetic billiard map, while the other component is not invariant. Nevertheless, the presence of this invariant boundary component suggests that an analogue of Theorem 1.1 may hold for magnetic billiards with convex planar tables.

4. Vanishing Lyapunov exponents characterize the integrable standard map

This section deals with random perturbations of the standard map and contains the proof of Theorem 1.2, obtained by combining Propositions 4.1 and 4.2.

The *standard map* is the one-parameter family of diffeomorphisms

$$g_K \colon \mathbb{T}^2 \to \mathbb{T}^2, \quad K \in \mathbb{R},$$

defined by

$$g_K(y_1, y_2) = (y_1 + y_2 + K \sin(2\pi y_1), y_2 + K \sin(2\pi y_1)) \bmod 1$$

for every $(y_1, y_2) \in \mathbb{T}^2$. A direct computation yields

$$\det Dg_K(y_1, y_2) \equiv 1, \quad \operatorname{tr} Dg_K(y_1, y_2) = 2 + 2\pi K \cos(2\pi y_1).$$

In particular, g_K preserves the normalized Riemannian volume m on $\mathbb{T}^2$. For $K = 0$, the standard map becomes the integrable map

$$g_0(y_1, y_2) = (y_1 + y_2,\ y_2).$$

Let $\lambda^+(\omega, y)$ denote the maximal Lyapunov exponent of the ν-random additive perturbation of g_K. In the next two propositions, we consider the cases when $\nu \ll m$ and when $\nu \perp m$.

Proposition 4.1. *Suppose $\nu \ll \mathrm{vol}$ with $B_\epsilon(0) \subseteq \mathrm{supp}\, \nu$ for some $\epsilon > 0$. Then $\lambda^+(\omega, y) = 0$ for $(\rho_\nu \times m)$-a.e. $(\omega, y) \in \Omega \times \mathbb{T}^2$ if and only if $K = 0$.*

Proof. If $K \neq 0$, there exist points $w, z \in \mathbb{T}^2$ such that

$$\mathrm{tr}\, Dg_K(w) > 2 \quad \text{and} \quad 0 < \mathrm{tr}\, Dg_K(z) < 2.$$

By Corollary 2.15, it follows that

$$\lambda^+(\omega, y) > 0 \quad \text{for } (\rho_\nu \times m)\text{-a.e. } (\omega, y) \in \Omega \times \mathbb{T}^2.$$

If $K = 0$, then

$$Dg_0 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Arguing exactly as in the proof of Lemma 3.3, we conclude that

$$\lambda^+(\omega, y) = 0 \quad \text{for } (\rho_\nu \times m)\text{-a.e. } (\omega, y) \in \Omega \times \mathbb{T}^2.$$

□

Proposition 4.2. *Suppose that*

$$\nu(E) = \int_0^{2\pi} \chi_E(0, y_2) h(y_2) dy_2$$

with $h \in L^1(dy_2)$ and $\{0\} \times [-\epsilon, \epsilon] \subseteq \mathrm{supp}\, \nu$ for some $\epsilon > 0$. Then $\lambda^+(\omega, y) = 0$ for $(\rho_\nu \times m)$-a.e. $(\omega, y) \in \Omega \times \mathbb{T}^2$ if and only if $K = 0$.

Proof. Recall that τ_x denotes the translation on $\mathbb{T}^2$ by $x \in \mathbb{T}^2$, and that $f_x = \tau_x \circ g_K$. Let $(0, a) \in \mathbb{T}^2$. A direct computation shows that

$$g_K \circ \tau_{(0,a)} = \tau_{(a,a)} \circ g_K.$$

As a consequence, for all $(0, a), (0, b) \in \mathbb{T}^2$, we have

$$f_{(0,b)} \circ f_{(0,a)} = \tau_{(0,b)} \circ g_K \circ \tau_{(0,a)} \circ g_K = \tau_{(a,a+b)} \circ g_K^2. \tag{4.1}$$

Let $\mathcal{I} : \mathbb{T}^2 \to \mathbb{T}^2$ be the coordinate-swapping map given by

$$\mathcal{I}(y_1, y_2) := (y_2, y_1), \quad y = (y_1, y_2) \in \mathbb{T}^2,$$

and define the map $\phi : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ by

$$\phi(x, y) = x + y + \mathcal{I}x, \quad x, y \in \mathbb{T}^2.$$

Let $x, y \in \mathrm{supp}\, \nu$. Since $\mathrm{supp}\, \nu \subset \{0\} \times S^1$, we may write $x = (0, a)$ and $y = (0, b)$ for some $a, b \in S^1$. Then

$$\phi(x, y) = (0, a) + (0, b) + (a, 0) = (a, a + b).$$

By identifying $\{0\} \times S^1$ with S^1, it follows that the restriction of ϕ to $\operatorname{supp} \nu \times \operatorname{supp} \nu$ coincides with the linear toral map

$$A : \mathbb{T}^2 \to \mathbb{T}^2, \quad A(a, b) = (a, a + b).$$

Let $\nu' = \phi_*(\nu \times \nu)$. Since ν is absolutely continuous with respect to Lebesgue measure on $\{0\} \times S^1$, the product measure $\nu \times \nu$ is absolutely continuous with respect to Lebesgue measure on $(\{0\} \times S^1)^2$. Since A is a linear automorphism of $\mathbb{T}^2$ with determinant 1, it follows that

$$\nu' \ll \mathrm{vol}\,.$$

Moreover, since $\operatorname{supp} \nu$ contains the interval $\{0\} \times [-\varepsilon, \varepsilon]$, the image of $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$ under A contains the ball $B_{\varepsilon/2}(0) \subset \mathbb{T}^2$. Hence,

$$B_{\varepsilon/2}(0) \subset \operatorname{supp} \nu'.$$

The previous observations allow us to conclude that ϕ is an isomorphism mod 0 between the probability spaces $(\mathbb{T}^2 \times \mathbb{T}^2, \mathcal{B} \times \mathcal{B}, \nu \times \nu)$ and $(\mathbb{T}^2, \mathcal{B}, \nu')$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{T}^2$.

Let $(\Omega', \Sigma', \rho_{\nu'})$ be the probability space given by the product of countably many copies of $(\mathbb{T}^2, \mathcal{B}, \nu')$. Note that the measurable spaces (Ω', Σ') and (Ω, Σ) coincide. Denote by $\sigma : \Omega \to \Omega$ and $\sigma' : \Omega' \to \Omega'$ the shift maps; these preserve ρ_ν and $\rho_{\nu'}$, respectively.

Define the map $\widetilde{\phi} : \Omega \to \Omega'$ by

$$(\widetilde{\phi}(\omega))_n = \phi(\omega_{2n}, \omega_{2n+1}), \quad \omega \in \Omega,\ n \in \mathbb{N}.$$

By construction of ν' and the fact that ϕ is an isomorphism mod 0 between $(\mathbb{T}^2 \times \mathbb{T}^2, \mathcal{B} \times \mathcal{B}, \nu \times \nu)$ and $(\mathbb{T}^2, \mathcal{B}, \nu')$, it follows that $\widetilde{\phi}$ is an isomorphism mod 0 between $(\Omega, \Sigma, \rho_\nu)$ and $(\Omega', \Sigma', \rho_{\nu'})$ such that

$$\widetilde{\phi} \circ \sigma^2 = \sigma' \circ \widetilde{\phi}.$$

Let $F : \Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$ be the ν-random additive perturbation of g_K, and let $G : \Omega' \times \mathbb{T}^2 \to \Omega' \times \mathbb{T}^2$ be the ν'-random additive perturbation of g_K^2. From (4.1), it follows that for every $\omega \in \Omega$ and every $n \geq 0$,

$$\begin{aligned} F_\omega^{2n} &= \tau_{\omega_{2n-1}} \circ g_K \circ \tau_{\omega_{2n-2}} \circ g_K \circ \cdots \circ \tau_{\omega_1} \circ g_K \circ \tau_{\omega_0} \circ g_K \\ &= \tau_{\phi(\omega_{2n-2}, \omega_{2n-1})} \circ g_K^2 \circ \cdots \circ \tau_{\phi(\omega_0, \omega_1)} \circ g_K^2 \\ &= G^n_{\widetilde{\phi}(\omega)}. \end{aligned} \tag{4.2}$$

Denote by λ_F^+ and λ_G^+ the maximal Lyapunov exponents of F and G, respectively. For every $(\omega, y) \in \Omega \times \mathbb{T}^2$, identity (4.2) implies that

$$\begin{aligned}\lambda_F^+(\omega, y) &= \limsup_{n\to+\infty} \frac{1}{2n} \log \|DF_\omega^{2n}(y)\| \\ &= \limsup_{n\to+\infty} \frac{1}{2n} \log \|DG^n_{\widetilde{\phi}(\omega)}(y)\| = \frac{1}{2}\lambda_G^+\left(\widetilde{\phi}(\omega), y\right).\end{aligned}$$

Therefore, $\lambda_F^+(\omega, y) > 0$ for $(\rho_\nu \times m)$-a.e. $(\omega, y) \in \Omega \times \mathbb{T}^2$ if and only if $\lambda_G^+(\omega', y) > 0$ for $(\rho_{\nu'} \times m)$-a.e. $(\omega', y) \in \Omega' \times \mathbb{T}^2$.

A direct computation yields

$$\begin{aligned}\operatorname{tr} Dg_K^2(y_1, y_2) = 2 &+ 4\pi K \cos(2\pi y_1) \\ &+ 4\pi K \cos\big(2\pi\big(y_1 + y_2 + K\sin(2\pi y_1)\big)\big) \\ &+ (2\pi K)^2 \cos(2\pi y_1)\, \cos\big(2\pi\big(y_1 + y_2 + K\sin(2\pi y_1)\big)\big).\end{aligned}$$

In particular, setting $y_1 = 1/4$, we obtain

$$\operatorname{tr} Dg_K^2\left(\tfrac{1}{4}, y_2\right) = 2 + 4\pi K \cos\big(2\pi y_2 + \tfrac{\pi}{2} + 2\pi K\big).$$

Moreover, for $k = 0$, we have

$$Dg_0^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The previous observations together with the properties of the probability measure ν' allow us to repeat verbatim the proof of Proposition 4.1 for g_K^2 in place of g_K, and conclude that $\lambda_G^+ = 0$ a.e. if and only if $K = 0$. □

The standard map g_K belongs to a broader class of toral maps $\mathcal{G}_V : \mathbb{T}^2 \to \mathbb{T}^2$ of the form

$$\mathcal{G}_V(y_1, y_2) = \big(y_1 + y_2 + V(y_1),\, y_2 + V(y_1)\big) \bmod 1,$$

where $V : S^1 \to \mathbb{R}$ is a C^1 function. For $V \equiv 0$, the map reduces to the integrable map

$$\mathcal{G}_0(y_1, y_2) = (y_1 + y_2, y_2) \bmod 1,$$

while choosing $V(y_1) = K\sin(2\pi y_1)$ yields the standard map g_K.

We have

$$D\mathcal{G}_V(y_1, y_2) = \begin{pmatrix} 1 + V'(y_1) & 1 \\ V'(y_1) & 1 \end{pmatrix}.$$

In particular,

$$\det D\mathcal{G}_V(y_1, y_2) = 1, \quad \operatorname{tr} D\mathcal{G}_V(y_1, y_2) = 2 + V'(y_1).$$

If V is constant, then $V' \equiv 0$ and

$$D\mathcal{G}_V \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If V is nonconstant, then $\max V > \min V$, and hence V' takes both positive and negative values. Consequently, $\operatorname{tr} D\mathcal{G}_V > 2$ at some points, and $0 < \operatorname{tr} D\mathcal{G}_V < 2$ at others.

These observations show that the arguments used in the proofs of Propositions 4.1 and 4.2 apply verbatim to the map $\mathcal{G}_V$. In particular, the conclusions of those propositions remain valid for $\mathcal{G}_V$, with the condition $K = 0$ replaced by the condition that V is constant.

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