Fast notes on symplectomorphisms

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Abstract

Some basic introductory fast notes on symplectic topology and geometry with emphasis in symplectomorphisms and Hamiltonian dynamics. To read this it is assumed some knowledge of differential geometry.

1 Introduction

It has been realised from the study of classical mechanics, variational calculus, geometrical optics, wave propagation, etc., the existence of a relevant set of transformations of the phase space of a dynamical system. This is a subset of the larger set of volume-preserving diffeomorphisms. Besides having the incompressibility property, those diffeomorphisms preserve a symplectic structure of the phase space, thus are called **symplectomorphisms**. They form a subgroup and present particular geometrical and topological properties and global invariants.

A **symplectic structure** ω defined on an even-dimensional smooth manifold M^{2d} is a closed non-degenerate differential 2-form (a more detailed account on this definition is contained in Section 2). The pair (M, ω) is called a **symplectic manifold**.

The symplectomorphisms preserve the symplectic form, thus also the naturally induced volume form $\omega^d = \omega \wedge \cdots \wedge \omega$. In this way, symplectomorphisms are volume-preserving and cannot have attractors (this is known for Hamiltonian flows by using Liouville's theorem – which connects to ergodic theory). The preservation of the symplectic structure gives rise to this and several more constraints on the admissible dynamical behaviour, as we shall see in the following.

The word symplectic was introduced by Weyl [11] to describe the finite-dimensional group of linear transformations preserving a non-degenerate skew-symmetric bilinear form. The name symplectic geometry was then used by Siegel [9] to study the geometry of that linear group. Nowadays

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this nomenclature is extended to non-linear symplectic manifolds and maps. As an additional remark, symplectic geometry can be regarded in a sense as a complexification of Riemannian geometry (cf. Arnol'd's *mathematical dream* of "simplectization, complexification and mathematical trinities" [3]). The matrix $\mathbb{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, an essential ingredient in mechanics, satisfies $\mathbb{J}^2 = -I$, an analogue of i in complex analysis.

Symplectic topology is more recent and aims to understand global symplectic phenomena. It has become an active research area of its own in, roughly, three main lines of development briefly summarised below (for more details and insights on results see e.g. [1, 2, 4, 8]):

- Variational methods and variational theory of capacities (Rabinowitz, Weinstein, Conley, Zehnder).
- Flexibility/rigidity, almost-complex structures associated to a symplectic form, theory of symplectic homology (Gromov, Floer).
- Globalisation of the classical idea of a generating function – related to variational formulation (Viterbo, Chaperon, Laudenbach, Sikorav).

To follow this notes it is assumed some knowledge of differential geometry, in particular calculus of differential forms (a good reference is [6]). The application of this language to Hamiltonian dynamics simplifies its study, in contrast with traditional notation. That is adequate if we write functions in terms of coordinates e.g. $H(q,p)$ when the coordinates are fixed. But when we change coordinates it is rather ambiguous to distinguish when $H(q',p')$ denotes the same function with new arguments, or is a different function of (q',p') with the same numerical value as H with respect to (q,p). The need to consider partial derivatives still increases the "entropy" on the reader's mind!

2 Symplectic structure

We say that a differential 2-form $\omega \in \Omega^2(M)$ on a C^∞-manifold M^{2d} is closed if $d\omega = 0$. This is a condition of geometrical nature.

Moreover, ω is non-degenerate if, for every $q \in M$ and $X \in T_qM$, $\omega(X,Y)(q) = 0$ implies $Y = 0$. This is the same as to say that $\omega^d \neq 0$. Thus, for every $q \in M$, the map

$$\begin{aligned} T_qM &\to T_q^*M \\ X(q) &\mapsto (\iota_X\omega)(q) := \omega(X,\cdot)(q) \end{aligned}$$

is an isomorphism. So, there is a one-to-one correspondence between 1-forms $\iota_X\omega \in T^*M$ and vector fields $X \in TM$, or simply between the tangent and the cotangent bundles (as in Riemannian geometry).

The conditions on the structure imply that M has to be even-dimensional and orientable.

Example 2.1 If $M = \mathbb{R}^{2d}$, the canonical symplectic form is given by

$$\omega_0 = \sum_{j=1}^{d} dx_j \wedge dy_j .$$

Notice that this form is exact since we can write it as $\omega = d\theta$ where

$$\theta = \sum_{j=1}^{d} y_j dx_j .$$

Choosing $v, w \in \mathbb{R}^{2d}$, ω_0 is calculated like

$$\begin{aligned} \omega_0(v,w) &= \sum_{j=1}^{d} dx_j \otimes dy_j(v,w) - dy_j \otimes dx_j(v,w) \\ &= \sum_{j=1}^{d} (v_j w_{d+j} - v_{d+j} w_j) \\ &= v^t \mathbb{J} w = \langle \mathbb{J} v, w \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard Riemannian metric on $\mathbb{R}^{2d}$. Section 7 contains more on this case. □

Example 2.2 The most natural example of a symplectic structure is given on the cotangent bundle T^*M of any given smooth manifold M. It is an exact form $\omega = d\theta$ where θ is related to the 1-forms of the sections of T^*M. We start by the coordinate-free definition and then present the local coordinates version.

A point in the cotangent bundle T^*M can be written as (q, σ_q), where q belongs to the manifold M and σ_q is a 1-form on M taken on q, i.e. $\sigma_q \in T_q^*M$. The main ideia is that, for each point in T^*M, we can choose θ to be essentially the 1-form σ_q itself. This is done via the natural projection π of T^*M on M (more precisely, its derivative), sending tangent vectors at (q, σ_q) to tangent vectors in the tangent space of M at q. Consider the projection of the cotangent bundle onto the manifold: $\pi \colon T^*M \to M, (q, \sigma_q) \mapsto q$. Its derivative at $(q, \sigma_q) \in T^*M$ is a linear map

$$d\pi(q, \sigma_q) \colon T_{(q,\sigma_q)} T^*M \to T_q M, \quad d\pi(q, \sigma_q)\,(\xi, \eta) = \xi .$$

The 1-form $\theta \colon TT^*M \to \mathbb{R}$ is then chosen to be at each point $(q, \sigma_q) \in T^*M$ given by $\theta_{(q,\sigma_q)} = \sigma_q \circ d\pi(q, \sigma_q)$. In what follows we show that the so-called canonical form on the cotangent bundle $\omega = d\theta$ is non-degenerate.

In local coordinates on T^*M, $(x(q), y(\sigma_q)) \in \mathbb{R}^{2d}$,

$$d\pi(q,\sigma_q)\,\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \quad \text{and} \quad d\pi(q,\sigma_q)\,\frac{\partial}{\partial y_j} = 0.$$

So, we can write the 1-form on M as $\sigma_q = \sum_j y_j dx_j$. Therefore, $\theta = \sum_j y_j dx_j$ and $\omega = \omega_0$ which is obviously non-degenerate. □

In the same way as before, we can identify a vector field X with the $(2d-1)$-form $\iota_X \omega^d$.

The identification of vector fields with forms is quite important because it allows us to describe vector fields that generate flows preserving the structure. So, the nature of the systems is fully contained in the choice of the form.

Another property arising from the definitions of forms is the following

Theorem 2.3 (Darboux) *Let ω be a symplectic or volume-preserving form on M and B a neighbourhood of $q \in M$. Then, there is a local diffeomorphism $\psi\colon B \to \mathbb{R}^{2d}$ such that $\psi^*\omega_0 = \omega$.*

Note that Darboux's theorem implies that there are no local symplectic or volume invariants. So, the theory of symplectic invariants and obstructions (see Section 3 for examples) are eminently global.

Two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) are said to be **symplectomorphic** if there is a diffeomorphism $f\colon M_1 \to M_2$ satisfying $f^*\omega_2 = \omega_1$.

There is also the concept of finite or infinite dimensional Poisson manifolds. The basic structure is the Poisson bracket on the space of functions rather than 2-forms. We will only mention that a Poisson manifold is symplectic by relating the symplectic structure with the Poisson bracket (cf. Appendix 14 of [2]).

3 Examples of symplectic phenomena

The following examples show that the behaviour of symplectomorphisms may be different from that of volume-preserving diffeomorphisms. The case of $\dim M = 2$ is different though, since there the group of area-preserving diffeomorphisms coincides with the one of symplectomorphisms.

Example 3.1 (Nonsqueezing and the symplectic camel) To understand the geometry of a map one usually starts by looking at the image of simple objects like balls. In the case of volume-preserving diffeomorphisms the ball is diffeomorphic to the image and both have the same volume. Gromov's nonsqueezing theorem (1985) proves that a standard ball cannot be symplectically embedded into a thin cylinder except for $\dim M = 2$. Thus there are volume-preserving diffeomorphisms that are not symplectic. Also, it shows that there is a basic property of the ball and the cylinder which is preserved by symplectomorphisms. These universal obstacles in the symplecting embedding led to the definition of Gromov's width and symplectic capacities.

A related problem consists of determining the possibility of continuously transforming by simplectomorphisms a ball (or a camel!) through a small "hole in a wall". Gromov's methods answered negatively for dimension 4 or higher. Again, it is possible to do so in dimension 2 as the answer is true for area-preserving diffeomorphisms. □

Example 3.2 (Symplectic fixed point theorems) Consider the symplectic 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ whose symplectic structure is inherited from $\mathbb{R}^2$, i.e. the standard area form. Let H be a time-independent Hamiltonian on $\mathbb{T}^2$ and ϕ the time-1 map of the associated flow (a symplectomorphism). Now, we want to count the number of fixed points of ϕ. We can start by looking at the critical points of H as the equation of motion is $(\dot{x}, \dot{y}) = \mathbb{J}\nabla H(x, y) = 0$. If H is sufficiently C^2-small, those are the only fixed points, so, we can count the number of fixed points by the critical points of H. It is at least two since $\mathbb{T}^2$ is compact and every non-constant function has distinct maximum and minimum. In fact, the minimum number of critical points, $\mathrm{Crit}(M)$, taken over all smooth functions on a compact manifold M is a topological invariant of M (using Morse theory of differential geometry). It is known that $\mathrm{Crit}(\mathbb{T}^2) = 3$.

For any compact symplectic manifold and an arbitrary large Hamiltonian flow whether depending on time or not, Arnol'd conjectured that the same kind of estimates on the number of fixed points remained true. This conjecture for $\mathbb{T}^2$ can be considered as a generalisation of the Poincaré-Birkhoff theorem (also known as Poincaré's last geometric theorem) on the existence of two fixed points for area-preserving twist maps (for short, twist maps) of the annulus.

The connection between fixed points of a mapping and critical points of the generating function is underneath the "duality" between twist maps and Frenkel-Kontorova models. The latter being one-dimensional discrete elastic chains of oscillators in a periodic potential with gradient-like dynamics (cf. [10]). □

4 Diffeomorphisms and vector fields

Consider a symplectic manifold (M, ω) and the induced volume form $v = \omega^d$ (sometimes called symplectic volume). Let us introduce the following notations:

- $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M,

- $\mathrm{Diff}_{\mathrm{vol}}(M, v) = \{\psi \in \mathrm{Diff}(M) \colon \psi^* v = v\}$, the subgroup of volume-preserving diffeomorphisms of M,

- $\mathrm{Symp}(M, \omega) = \{\psi \in \mathrm{Diff}(M) \colon \psi^* \omega = \omega\}$, the subgroup of symplectomorphisms of (M, ω).

Symplectomorphisms can also be defined as being the ones among all diffeomorphisms that preserve the Poisson bracket: for any $f, g \in C^\infty(M, \mathbb{R})$,

$$\{f, g\} \circ \psi = \{f \circ \psi, g \circ \psi\} \quad \text{iff} \quad \psi \in \mathrm{Symp}(M, \omega).$$

Let $\mathcal{X}(M)$ stand for the set of every smooth vector fields on M. Vector fields can be regarded as differential operators on functions. If $X \in \mathcal{X}(M)$ and $f \in C^1(M, \mathbb{R})$, we have $X f = df(X) = \mathcal{L}_X(f)$, where $\mathcal{L}_X$ is the Lie derivative with respect to X. When evaluated at $q \in M$, this is the directional derivative of f at q along $X(q)$.

We can classify the elements of $\mathcal{X}(M)$ by the following definitions:

- $X \in \mathcal{X}(M)$ is a **symplectic vector field** iff $\iota_X \omega$ is closed, i.e. $d(\iota_X \omega) = 0$ or equivalently by Cartan's formula $\mathcal{L}_X(\omega) = d(\iota_X \omega) + \iota_X(d\omega) = 0$. This means that the symplectic form is constant along the flow of X. We therefore write $X \in \mathcal{X}(M, \omega)$.

- $X \in \mathcal{X}(M)$ is a **Hamiltonian vector field** iff $\iota_X \omega$ is exact, i.e. $\iota_X \omega = dH$ for some primitive function $H \in C^1(M, \mathbb{R})$ called the **Hamiltonian function**. In this case a function identifies (generates) a vector field and we denote X by X_H.

It is clear that Hamiltonian implies symplectic, because exact implies closed. Locally, any symplectic vector field is Hamiltonian. Globally, whenever the (de Rham) first cohomology group of M vanishes: $H^1(M) = 0$ (i.e. all the closed 1-forms are exact), symplectic vector fields are Hamiltonian.

Exercise 4.1 Let $M = \mathbb{T}^2$ and $\omega = dx \wedge dy$. Show that the vector fields $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y}$ are symplectic but not Hamiltonian.

Proposition 4.1 *Let M be a closed (compact and without boundary) manifold, and the 1-parameter family $\psi^t \in \mathrm{Diff}(M)$ generated by the 1-family of vector fields $X^t \in \mathcal{X}(M)$, i.e.*

$$\frac{d}{dt}\psi^t = X^t \circ \psi^t, \quad \psi^0 = \mathrm{Id}.$$

Then:

- $\psi^t \in \mathrm{Symp}(M, \omega)$ *iff* $X^t \in \mathcal{X}(M, \omega)$. *So, symplectic vector fields generate symplectomorphisms. (Notice that* ψ^t *is a isotopy to the identity.)*

- *If Y, Z are symplectic vector fields, their commutator $[Y, Z] = \mathcal{L}_Y(Z)$ is a Hamiltonian vector field with Hamiltonian function $\omega(Y, Z)$. Thus, we write $[Y, Z] = X_{\omega(Y,Z)}$.*

Remark 4.2 From the proposition above, given the Hamiltonian functions $f, g \in C^\infty(M, \mathbb{R})$ and their respective Hamiltonian vector fields X_f, X_g, we have $\omega(X_f, X_g) = \{f, g\}$ and $X_{\{f,g\}} = [X_f, X_g]$. Therefore, a Poisson manifold is symplectic.

Any smooth 1-parameter family ψ^t in $\mathrm{Symp}(M, \omega)$, $t \in [0, 1]$, with $\psi^0 = \mathrm{Id}$ is called a **symplectic isotopy** on M. This means that ψ^1 is symplectic isotopic to the identity. We can show that such isotopies are always generated by a vector field as in Proposition 4.1.

- A symplectic isotopy is generated by a unique 1-family of symplectic vector fields $X^t \in \mathcal{X}(M, \omega)$.
- If $\iota_{X^t}\omega$ is exact, we can find a smooth 1-family of time-dependent Hamiltonians $H^t \in C^\infty(M, \mathbb{R})$ such that $X^t = X_{H^t}$. So, ψ^t is a **Hamiltonian isotopy**.

In particular, if M is simply connected, every symplectic isotopy is Hamiltonian.

A symplectomorphism is Hamiltonian if it is Hamiltonian-isotopic to the identity. The space of **Hamiltonian symplectomorphisms** is denoted by $\mathrm{Ham}(M, \omega)$ and it is a normal subgroup of $\mathrm{Symp}(M, \omega)$. The corresponding Lie algebra is the algebra of all Hamiltonian vector fields.

Exercise 4.2 Show that for $M = \mathbb{T}^2$ and the standard area form $dx \wedge dy$, the symplectomorphisms isotopic to the identity preserve the "centre of gravity", i.e. they are of the form:

$$\psi(x, y) = (x + f(x, y), y + g(x, y)), \quad \text{where} \quad \int_{\mathbb{T}^2} f = \int_{\mathbb{T}^2} g = 0.$$

There is an interesting relation between the Lie groups of diffeomorphisms and the corresponding Lie algebras with respect to the commutator $[\cdot, \cdot]$ for vector fields, see Table 4. It is opportune to remark also that $C^\infty(M, \mathbb{R})$ is a Lie algebra under the Poisson bracket. Furthermore, the transformation $H \mapsto X_H$ from $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ to the space of Hamiltonian vector fields with the commutator operator, is a morphism of Lie algebras.

5 Hamiltonian dynamics

A **Hamiltonian system** is a triple (M, ω, H) where (M, ω) is a symplectic manifold and $H \in C^\infty(M, \mathbb{R})$ is the Hamiltonian function. The vector field

Table 1: Comparing the classes of diffeomorphisms of M.

Lie group	Lie algebra of vector fields	
$\mathrm{Diff}(M)$	$\mathcal{X}(M)$	differential geometry
$\mathrm{Diff}_{\mathrm{vol}}(M, v)$	divergence-free	incompressible hydrodynamics
$\mathrm{Symp}(M, \omega)$	locally Hamiltonian	symplectic geometry
$\mathrm{Ham}(M, \omega)$	Hamiltonian	Hamiltonian dynamics

$X_H \in \mathcal{X}(M, \omega)$ determined by the identity $\iota_{X_H}\omega = dH$ is the Hamiltonian vector field associated to H. (Notice that H is not a function of time t. This can be always the case by considering the extended manifold $M' = M \times \mathbb{R}$ and a point on it is (q, t).)

The vector field X_H generates the **Hamiltonian flow**, i.e. a smooth 1-parameter group of Hamiltonian symplectomorphisms $\phi_H^t \in \mathrm{Ham}(M, \omega)$ such that

$$\frac{d}{dt}\phi_H^t = X_H \circ \phi_H^t, \quad \phi_H^0 = \mathrm{Id}.$$

In these conditions, ϕ_H^t is an isotopy for a time-independent Hamiltonian. We remark also that

$$\mathcal{L}_{X_H}(H) = X_H\, H = dH(X_H) = (\iota_{X_H}\omega)X_H = \omega(X_H, X_H) = 0.$$

So, the Hamiltonian vector field X_H is tangent to the level sets of H.

A function $f \in C^\infty(M, \mathbb{R})$ such that $\{f, H\} = \frac{d}{dt}(f \circ \phi_H^t) = 0$ is called an **integral of motion**. Classical mechanics is all about finding integrals of motion and the flow ϕ_H^t from a given function $H = T + V$ corresponding to a certain physical problem with kinetic energy T and under a potential energy V.

Exercise 5.1 (Harmonic oscillator) The following Hamiltonian on $T^*\mathbb{R}^d = \mathbb{R}^{2d}$ gives a flow on the sphere and generalises the classical two-dimensional harmonic oscillator:

$$H(x, y) = \frac{1}{2}\|(x, y)\|_a^2,$$

where it is used the weighted norm $\|(x, y)\|_a = [\sum_{j=1}^d a_j(x_j^2 + y_j^2)]^{1/2}$, $a_j > 0$. Find the flow and prove integrability. Find all periodic orbits for a level set $H^{-1}(h)$, $h > 0$. **Hint:** Identify $\mathbb{R}^{2d}$ to $\mathbb{C}^d$ by $z = (z_1, \dots, z_d) = (x_1 + iy_1, \dots, x_d + iy_d)$. The symplectic matrix $\mathbb{J}$ thus acts by multiplication with i and $X_H(z) = -iz$.

In the following, we use the pull-back of a vector field X that is defined by $\psi^*X(q) = d\psi(q)^{-1}\, X(\psi(q))$, for any $q \in M$. As an example, $\phi_H^{t*}\, X_H = X_H$, which is equivalent to say that the Hamiltonian flow is generated by the vector field X_H.

Proposition 5.1 *Let (M, ω, H) be a Hamiltonian system.*

- *The level sets of H are invariant codimension-1 submanifolds of M under the Hamiltonian flow (conservation of energy).*
- *$X_{H\circ\psi} = \psi^* X_H$, with $\psi \in \mathrm{Symp}(M, \omega)$. In other words, $\psi^* X_H$ is the Hamiltonian vector field for the Hamiltonian function $H \circ \psi$, so ψ is what in the classical literature is called a canonical transformation.*

Notice that, if H, H' are Hamiltonian functions on the same symplectic manifold, then

$$\begin{aligned} \mathcal{L}_{X_H}(H') &= dH'(X_H) = \omega(X_{H'}, X_H) \\ &= -\omega(X_H, X_{H'}) = -dH(X_{H'}) \\ &= -\mathcal{L}_{X_{H'}}(H). \end{aligned}$$

This is a somewhat interesting property of Hamiltonian systems.

If ω is integrated along two-dimensional surfaces in M, we obtain the so-called Poincaré integral invariant, which is preserved by the Hamiltonian flow.

A good source of information on these topics are [2, 7].

6 Integrability

A Hamiltonian system (M, ω, H) is **completely integrable** iff there exists integrals of motion $f_1, \ldots, f_d$ in involution, i.e. $\{f_j, f_k\} = 0$.

The Liouville-Arnol'd theorem states that integrable systems can be written in action-angle variables in which there are explicit formulae for their solutions. If the level set (a d-dimensional submanifold of M) of the integrals of motion f_j is compact and connected, it is diffeomorphic to $\mathbb{T}^d$. Moreover, in the neighbourhood of every such invariant torus one can find new coordinates called the action-angle variables $(\theta, I) \in \mathbb{T}^d \times \mathbb{R}^d$, obtained from the initial variables $(q, p) = \psi(\theta, I)$ such that the Hamiltonian flow is now simply given by

$$(\dot{\theta}, \dot{I}) = \left(\partial H'/\partial I, 0\right).$$

In other words, the new Hamiltonian function $H' = H \circ \psi$ depends only on I and the phase space is foliated by invariant tori $\{I = \text{const}\}$ on which the flow is linear. This shows that the dynamics of integrable systems are extremely simple.

The invariant level sets are also interesting from the symplectic point of view. The symplectic form ω vanishes on them and such submanifolds are called Lagrangian. They play an important role in symplectic topology.

The type of dynamical behaviour exhibited by integrable systems (that are very rare) is highly exceptional. An arbitrarily small perturbation may destroy many of these invariant tori. On the other hand, if the frequency vector $\omega = \partial H'(I_0)/\partial I$ satisfies a diophantine condition (vector with rationally independent coordinates and not "well approximated" by others with rationally dependent coordinates) and $\partial^2 H'/\partial I^2$ is non-singular, then the corresponding invariant torus $\{I = I_0\}$ survives slightly deformed under sufficiently small perturbations. This is the content of KAM theory (a rather complete review on the subject is in [5]).

7 The fundamental case: $\mathbb{R}^{2d}$

The standard symplectic form on $\mathbb{R}^{2d}$ can be thought of a skew-symmetric bilinear form on the tangent space $T_z\mathbb{R}^{2d} = \mathbb{R}^{2d}$, $z = (x, y) \in \mathbb{R}^{2d}$: $\omega_0 = \sum_{j=1}^{d} dx_j \wedge dy_j$.

There is a special group inside $\mathrm{Symp}(\mathbb{R}^{2d}, \omega_0)$ which is formed by the symplectic linear maps and denoted by $\mathrm{Sp}(2d, \mathbb{R})$. Its elements can be represented by the associated matrix with respect to the Euclidean canonical base, and can be defined by the conditions: $\Psi \in \mathrm{Sp}(2d, \mathbb{R})$ iff $\Psi^t \mathbb{J} \Psi = \mathbb{J}$ iff $\Psi^* \omega_0 = \omega_0$. Hence, as it preserves the volume form as well, $\det(\Psi) = 1$, and $\mathrm{Sp}(2d, \mathbb{R}) \subset SL(2d, \mathbb{R})$.

Exercise 7.1 Find an element of $SL(4, \mathbb{R})$ not in $\mathrm{Sp}(4, \mathbb{R})$. Show that $\mathrm{Sp}(2, \mathbb{R}) = SL(2, \mathbb{R})$.

We can generalise the group of linear symplectomorphisms on $\mathbb{R}^{2d}$ to an arbitrary symplectic vector space (V, ω). We denote it by $\mathrm{Sp}(V, \omega)$ and its elements are linear isomorphisms $\Psi \colon V \to V$ which preserve ω. Nevertheless, from a result that states that all symplectic vector spaces of the same dimension are linearly symplectomorphic (theorem 2.3 in [8]), we can restrict our attention to $(\mathbb{R}^{2d}, \omega_0)$.

The phase space using the action-angle variables discussed in Section 6 is $\mathbb{T}^d \times \mathbb{R}^d$. Considering the universal cover of the d-torus, we can lift this manifold to $\mathbb{R}^{2d}$ preserving the periodicity on the angle variable. This is indeed the standard way of treating the problem.

The above considerations together with Darboux's theorem resume the arguments to consider $(\mathbb{R}^{2d}, \omega_0)$ as a paradigm for local symplectic manifolds and linear symplectic spaces, worth studying in detail. An important fact is that the structure ω_0 is essentially unchanged by scalar multiplication. For $\lambda \in \mathbb{R}$, the linear map on $\mathbb{R}^{2d}$ given by $\phi_\lambda(x) = \lambda x$ acts on ω_0 by a simple rescaling, $\phi_\lambda^* \omega_0 = \lambda^2 \omega_0$. Since the global symplectic structures of interest to us are invariant under rescaling, they are a reflection of what happens inside small pieces of Euclidean space. Note also that the derivative at 0 of a diffeomorphism ϕ satisfying $\phi(0) = 0$ is the limit of rescalings:

$$d\phi(0)\, v = \lim_{t \to 0} \frac{\phi(tv)}{t}, \quad v \in \mathbb{R}^{2d}.$$

Thus one can think of the derivative as being given geometrically by looking at what happens on smaller and smaller pieces of the manifold. By this kind of zoomings and renormalisations, the local (or linear) theory may be thought of as the limit of the global (nonlinear) theory.

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