DYNAMICS OF DIFFERENTIAL EQUATIONS

RANDOM EXERCISES FOR SUPERVISIONS

(1) Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ continuous. Prove that if there is a continuous function $\lambda\colon \mathbb{R} \to \mathbb{R}$ such that

$$x \cdot f(t,x) \leq -\lambda(t) x \cdot x$$

then all the solutions of $\dot{x} = f(t,x)$ are global.

(2) Consider the initial value problem

$$\begin{cases} \ddot{z} + \alpha(z,\dot{z})\dot{z} + \beta(z) = u(t) \\ z(0) = \xi, \dot{z}(0) = \eta \end{cases}$$

where $\alpha\colon \mathbb{R}^2 \to \mathbb{R}$ and $\beta\colon \mathbb{R} \to \mathbb{R}$ are C^1 and satisfy $\alpha(z,y) \geq 0$ and $z\beta(z) \geq 0$, for all $z, y \in \mathbb{R}$. Show that exists one and only one solution of the problem and that can be defined in $[0, +\infty)$.

(3) Discuss the asymptotic stability of the solutions to the following system of equations:
   (a) $\dot{x} = -x(1-x)$
   (b) $\ddot{x} + x = 0$
   (c) $\ddot{x} + \frac{1}{2}\left[x^2 + \sqrt{x^4 + 4\dot{x}^2}\right] x = 0$

(4) Discuss the phase portrait of the equations
   (a)
$$\begin{cases} \dot{x} = y(y^2 - x^2) \\ \dot{y} = -x(y^2 - x^2) \end{cases}$$
   (b)
$$\begin{cases} \dot{x} = yz \\ \dot{y} = -xz \\ \dot{z} = xy \end{cases}$$

(5) Show that, if $P(x,y)$ and $Q(x,y)$ are two limited functions of class C^1, each one of the planar vector fields defined by

$$\begin{cases} \dot{x} = x^3 + P(x,y) \\ \dot{y} = y^3 + Q(x,y) \end{cases} \qquad \begin{cases} \dot{x} = -y + P(x,y) \\ \dot{y} = x + Q(x,y) \end{cases}$$

has at least one equilibrium point.

(6) Let $\ddot{x} = -\nabla U(x)$, $U \in C^2$, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, and $x_0 \in \mathbb{R}^n$ such that $U(x_0) = 0$ and x_0 is a local minimum for the potential $U = U(x)$. Prove that the point $(x, \dot{x}) = (x_0, 0) \in \mathbb{R}^{2n}$ is a stable equilibrium point.

(7) Consider the following ordinary differential equation

$$\ddot{x} + \left[(x^2 + \dot{x}^2)^2 - 3\alpha(x^2 + \dot{x}^2) + 2\alpha^2\right]\dot{x} + \frac{x}{2} = 0$$

(a) Show that the system is dissipative for all $\alpha \in \mathbb{R}$.
(b) Show that the system has a unique equilibrium point that is hyperbolic and stable for $\alpha \neq 0$.
(c) Show that for $\alpha > 0$ the system has two periodic orbits.
(d) Draw a sketch of the phase portrait for the different values of α.

(8) Consider the system of ODEs

$$\begin{cases} \dot{x} = y \\ \dot{y} = (\alpha - 4)x - y + 6x^2 - 2x^3 \end{cases}$$

(a) Show that this is a dissipative dynamical system for all $\alpha \in \mathbb{R}$.
(b) Compute the equilibria and its stability for $\alpha \neq 4$.
(c) Sketch the phase portrait for different values of α.

(9) Consider the system ruled by

$$\ddot{x} + f(x, \dot{x})\dot{x} + x = 0$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = \sin^2\left(\frac{\pi}{x^2+y^2}\right)$ for $(x, y) \neq 0$ and $f(0, 0) = 0$.
(a) Sketch the phase portrait, stating the equilibria, periodic orbits, etc.
(b) Classify the origin regarding its stability.
(c) Show that the system is dissipative and determine its global atractor.

Hint: Consider transforming to polar coordinates.

(10) Let be the following system of differential equations

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} - \lambda H \frac{\partial H}{\partial x} \\ \dot{y} = -\frac{\partial H}{\partial x} - \lambda H \frac{\partial H}{\partial y} \end{cases}$$

where $H(x, y) = y^2 - 2x^2 + x^4$, and $\lambda \in \mathbb{R}$.
(a) Show that for $\lambda = 0$ the system is conservative (Hamiltonian system of total energy H).
(b) Compute the equilibria, the periodic and homoclinic orbits, and sketch the phase portrait. Classify the equilibria regarding their stability.
(c) Redo the last question for $\lambda \neq 0$.
(d) Show that for $\lambda > 0$ the system is dissipative. Determine the ω-limit set of the points $(1/2, 0)$, $(-1/2, 0)$ and $(1, 2)$.

(11) Consider the following system

$$\begin{cases} \dot{x} = x(1 - kx) - y(1 - e^{-2x}) \\ \dot{y} = y(1 - e^{1-x}) \end{cases}$$

$k = e^{-2} = 0.135\ldots$.

(a) Show that the set $Q = \{(x, y) \in \mathbb{R}^2 \colon x \geq 0, y \geq 0\}$ is invariant for the local dynamical system corresponding to this system of ODE's.
(b) Compute the equilibria and sketch the directions of the vector field over the positive semi-axis.
(c) Show that any positive orbit in Q is global, i.e. it is defined for all $t \geq 0$.
(d) Show that the orbits with initial condition (x_0, y_0), $x_0 > 1$ and y_0 sufficiently large, enter in the region $R = \{(x, y) \in \mathbb{R}^2 \colon 0 < x < 1, y > 0\}$. Justify that the system is therefore dissipative in Q.
(e) Show that the equilibrium point inside the interior of Q is unstable.
(f) Show that there is a periodic orbit and sketch the phase portrait in Q.

(12) Consider the system based on a model of an impulse propagation in a biological system ($\mu > 0$)

$$\begin{cases} \dot{x} = \mu x - x^3 - y \\ \dot{y} = x - y \end{cases}$$

(a) Compute μ_0 such that for $\mu > \mu_0$ the system has three equilibria exactly. Sketch the directions of the vector field in $\mathbb{R}^2$.
(b) Deduce the stability of those equilibria.
(c) For each fixed μ prove that for all c sufficiently large, $Q_c = \{(x, y) \in \mathbb{R}^2 \colon |x| \leq c, |y| \leq c\}$ is positively invariant for the system.
(d) Show that there are no periodic orbits and conclude that it is a dissipative system.
Hint: Consider $(x, y) = F(x) - x^2 + xy - \frac{1}{2}y^2$, where $F'(x) = \mu x - x^3$, and determine the behaviour of V in the orbits of the system.
(e) Apply the Poincaré-Bendixson theorem to prove that for $\mu > \mu_0$ there exist two heteroclinic orbits from the origin to the other two equilibria. Sketch the phase portrait.

(13) Consider the system of ODEs

$$\begin{cases} \dot{x} = y^2 - x^4 \\ \dot{y} = -y - x^2 + x^3 \end{cases}$$

(a) Determine the dimensions of the stable, unstable and central manifolds of the origin.
(b) Determine approximations of the central manifold and of the flux over it. Decide about the stability of the origin.

(14) Consider the system of ODEs

$$\begin{cases} \dot{x} = -xz - yz \\ \dot{y} = -4y + xz + yz \\ \dot{z} = -z + x^2 - y^2 \end{cases}$$

(a) Compute the equilibria and the dimensions of the stable, unstable and central manifolds of the origin.

(b) Compute aproximations of the local central manifold and of the flux over it, in order to decide about the stability of the origin.