Renormalisation scheme for vector fields on $\mathbb{T}^2$ with a diophantine frequency

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April, 2001

Abstract

We construct a rigorous renormalisation scheme for analytic vector fields on $\mathbb{T}^2$ of Poincaré type. We show that iterating this procedure there is convergence to a limit set with a "Gauss map" dynamics on it, related to the continued fraction expansion of the slope of the frequencies. This is valid for diophantine frequency vectors.

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1 Introduction

A renormalisation operator $\mathcal{R}$ is defined for analytic vector fields on the torus of dimension two, associated with a frequency vector. This approach is based in [10] for Hamiltonian functions, and in [12] for flows on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 2$. The vector fields considered here are of Poincaré type, i.e. there is a classification by a unique winding ratio on which $\mathcal{R}$ acts as the Gauss map. The slope α of the winding ratio is mapped by the shift of its continued fraction expansion. All vector fields corresponding to non-zero α are renormalisable, and those with irrational α are infinitely renormalisable. We extend a result in [12] based on [10] for diophantine winding ratios.

The renormalisation group technique has been used for some time applied to various problems in dynamical systems (e.g. [7, 15, 17]). Roughly, it analyses systems on longer time scales and smaller spacial scales, generating a new system. By iterating this we may get convergence to a limiting behaviour, preferably simple, e.g. a fixed point. In such a case, we get self-similarity, which can be non-trivial depending on the problem.

We show here that an orbit by $\mathcal{R}$ of a constant vector field with diophantine winding ratio, attracts all the nearby orbits in the same homotopy class. This can be applied to other flows on domains with an extra vertical dimension. Also, it might be used for two-degrees of freedom Hamiltonian systems, in order to extend [10]. The allowed set of frequencies is smaller than the one obtained by KAM theory. Nevertheless, this is not a real disadvantage of our method since it is of equal full Lebesgue measure.

Our procedure can be related to the one followed by MacKay [13], and Chandre and Moussa [5] for an approximate renormalisation scheme in the framework of the breakup of invariant tori in Hamiltonian systems. The renormalisation studied by those authors, besides truncating the Fourier modes of the Hamiltonians, does not consider perturbations depending on the action variable. That would have implied a change in the Hamiltonian vector fields, namely the angle time derivative. Thus, it does not have the unstable direction that appears in our systems corresponding to perturbations non-collinear to ω.

In Section 2, we include the main idea for the construction of the linear part of the renormalisation, along with the main properties of the continued fraction expansion of an irrational number. Then, in Section 3 we introduce the space of vector fields of our interest. We define a one-step operator $\mathcal{R}$ in Section 4 and show one of its statements in Section 7. It follows Section 5 with more known results of number theory, this time on the diophantine condition. The main convergence theorem for diophantine vectors is presented in Section 6, along with a proof in Section 8.

2 Change of Basis for a Generic Frequency and the Continued Fraction Expansion

It was shown in [12] (using results from [10]) that for a quadratic irrational slope of a two-dimensional vector ω, the renormalisation operator therein constructed is well-defined since its linear part exists. More precisely, we can find a hyperbolic matrix T in $GL(2,\mathbb{Z})$ for which ω is an eigenvector. For other vectors one needs to construct a different operator. In fact, for each iteration of the renormalisation we will consider a different transformation, determined by the continued fraction expansion of α.

Consider the constant vector field ω on the universal cover $\mathbb{R}^2$, with slope $\alpha = \omega_2/\omega_1$, assuming that $\omega_1 \neq 0$. The main idea is to perform a change of basis enlarging the region of $\mathbb{R}^2$ around the orbits, and a scaling of time.

In the case $\alpha > 1$, we choose the new basis $\{(1,1),(0,1)\}$, that decreases ("D") the slope of the transformed vector: $\alpha' = \alpha - 1$. On the other hand, if $0 < \alpha < 1$, one simply swaps ("S") the coordinates, i.e. we choose $\{(0,1),(1,0)\}$ thus inverting the slope: $\alpha' = 1/\alpha$. The change of basis matrices are then $D = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, respectively. Note also that $D^a = \left[\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right]$, $a \in \mathbb{Z}$ and $S^{-1} = S$. Finally, if $\alpha < 0$, we recover one of the previous cases using $V = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = V^{-1}$.

Recall that the matrices D, S and V generate $GL(2,\mathbb{Z})$. This group acts on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ as $\alpha \mapsto (c + d\alpha)/(a + b\alpha)$ with $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2,\mathbb{Z})$. Each of the matrices D, S and V correspond to a transformation on $\bar{\mathbb{R}}$ (in fact their inverses since those are the direct transformations to the new basis): D and S as above and $V \colon \alpha' = -\alpha$.

The *continued fraction expansion* of $\alpha \in \mathbb{R}$, is given by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}},$$

or simply $\alpha = [a_0, a_1, a_2, \dots]$, with $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{Z}^+$, $n \geq 1$. It is an infinite sequence if and only if α is irrational [11]. We assume in the following that that is case and $\alpha > 0$. In these conditions, the continued fraction expansion is a one-to-one map.

Let $[x] = \max\{k \in \mathbb{Z} \colon k \leq x\}$ be the integer part of $x \in \mathbb{R}$ and $\{x\} = x - [x]$ the fractional one. The Gauss map is given by

$$G \colon x \mapsto \left\{ \frac{1}{x} \right\}, \quad x > 0.$$

Writing $x_n = G(x_{n-1})$, $n \geq 1$, with $x_0 = \{\alpha\}$, one obtains the above coefficients a_n from the recurrence relation $a_0 = [\alpha]$, $a_n = [1/x_{n-1}]$, $n \geq 1$.

Note that $\alpha_n = a_n + x_n$, $n \geq 0$, where $\alpha_n = [a_n, a_{n+1}, \dots]$, hence $\alpha_{n+1} = 1/\{\alpha_n\} = 1/x_n$. Define also

$$\beta_n = \prod_{i=0}^{n} x_i = \prod_{i=0}^{n} \frac{1}{\alpha_{i+1}}. \tag{1}$$

If $x_k \geq \gamma^{-1}$ for some $k = 0, \dots, n-1$, where $\gamma = \frac{1+\sqrt{5}}{2}$, then, letting $m = x_k^{-1} - x_{k+1} \geq 1$, $x_k x_{k+1} = 1 - m x_k \leq 1 - x_k \leq 1 - \gamma^{-1} = \gamma^{-2}$. On the other hand, if $x_k \leq \gamma^{-1}$, then $x_{k+1} \geq \gamma^{-1}$, and the product is again less or equal to γ^{-2}. Hence,

$$\beta_n \leq \gamma^{-n}, \qquad \frac{\beta_n}{\beta_{j-1}} \leq \gamma^{-(n-j)}, \quad 0 \leq j \leq n. \tag{2}$$

The quadratic irrationals (roots of a quadratic polynomial over $\mathbb{Q}$) have eventually periodic continued fraction expansion [11], e.g. $\gamma = [1,1,1,\dots]$ and $\sqrt{2} = [1,2,2,\dots]$.

The action of D^{a_0} on α is the elimination of the first coefficient a_0 of the continued fraction expansion, $[a_0, a_1, a_2, \dots] \mapsto [0, a_1, a_2, \dots]$. The transformation S acts on sequences of the form $[0, a_1, a_2, \dots]$ giving $[a_1, a_2, \dots]$. The composition is simply the shift of the coefficients.

We are looking for a specific type of linear coordinate changes of the torus. That is, it has to correspond to a matrix capable of improving the analyticity, as will be seen in the following sections (see also [12]). So, it has to be a hyperbolic matrix in $GL(2,\mathbb{Z})$, with an unstable direction close enough to the subspace spanned by ω. It turns out that we can choose the above shift of coefficients. The complete linear transformation is then

$$T_a = D^a S = \left[\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right] \in GL(2,\mathbb{Z}),$$

where $a = [\alpha]$. If $a \geq 1$, this is a hyperbolic matrix with two real eigenvalues: one inside and the other outside the unit circle. More specifically,

$$\lambda = \frac{a + \sqrt{a^2+4}}{2} \geq \gamma$$

and $-1/\lambda$ are the eigenvalues. The corresponding unstable and stable eigenvectors are $(1, \lambda)$ and $(1, -1/\lambda)$.

Now, it is useful to recall some properties of the continued fraction expansion, since we have associated it with the change of basis to be used in the construction of the renormalisation. We can write the above continued fraction expansion procedure in terms of the "convergent" matrices

$$P_n = \begin{bmatrix} q_{n-1} & p_{n-1} \\ q_n & p_n \end{bmatrix}, \quad n \geq 0, \qquad \text{and } P_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

using the recurrence formula $P_{n+1} = T_{n+1}P_n$, where $T_n = T_{a_n}$, and

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1}, \\ q_{n+1} &= a_{n+1}q_n + q_{n-1}, \\ \beta_{n-2} &= a_n\beta_{n-1} + \beta_n, \end{aligned} \tag{3}$$

assuming $\beta_{-1} = 1$. The last equality is simply derived from rewriting $\beta_{n-2} = \alpha_n\beta_{n-1}$. Notice that $T^*_{n+1} = (P^*_n)^{-1}P^*_{n+1}$ and $P_n = T_n \cdots T_0$, where T^* stands for the transpose matrix of T.

The continued fraction expansion can be obtained in only one step by a change of basis to $\{(q_{n-1}, p_{n-1}), (q_n, p_n)\}$, with the corresponding matrix being P^*_n. Thus, $(1, \alpha_{n+1}) = (q_{n-1} - \alpha q_n)P^*_n(1, \alpha)$. So,

$$\alpha = \frac{p_{n-1} + p_n\alpha_{n+1}}{q_{n-1} + q_n\alpha_{n+1}}, \tag{4}$$

which implies $x_n = -(q_n\alpha - p_n)/(q_{n-1} - p_{n-1})$ and, by (1),

$$\beta_n = (-1)^n(\alpha q_n - p_n).$$

Again, from (4) using $(n+1)$ instead of n, $|\beta_n| = 1/(q_{n+1} + q_n x_{n+1})$. Hence,

$$\frac{1}{2q_{n+1}} < \beta_n < \frac{1}{q_{n+1}}. \tag{5}$$

It is easy to see that $\det P_n = (-1)^{n+1}$ and each line of the matrices $P_n \in GL(2, \mathbb{Z})$ gives the convergents $p_n/q_n = [a_0, \dots, a_n] \to \alpha$, as $n \to \infty$. This sequence verifies the best diophantine approximation property, $|p_n - \alpha q_n| < |p - \alpha q|$, for any integer p and $0 < q < q_n$ ([8] - Theorem 182). Also, if $|\alpha - p/q| < 1/2q^2$, then p/q is a convergent of α ([8] – Theorem 184). Using the fact that $|\det P_n| = 1$, it is clear that the convergents verify $|p_n/q_n - p_{n+1}/q_{n+1}| = 1/q_n q_{n+1}$. Then,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}. \tag{6}$$

Denote the product of the first $(n+1)$ continued fraction coefficients of an irrational α by $A_n = \prod_{i=0}^n a_i$. Rewriting (3) one obtains $q_{n+1}/q_n = a_{n+1} + 1/(q_n/q_{n-1})$. By induction it can be proved that the following is true:

$$A_n \le q_n \le A_n \prod_{i=1}^{n}(1 + 1/a_i a_{i-1}).$$

Similarly, we define by $\tilde{A}_n$ the product of the $(n+1)$ first numbers $\alpha_i = [a_i, a_{i+2}, \dots]$:

$$\tilde{A}_n = \prod_{i=0}^{n} \alpha_i.$$

This is related trivially to β_n by the formula $\tilde{A}_{n+1} = \alpha_0/\beta_n$. So, from (3), $\tilde{A}_{n-1}^{-1} = a_0\tilde{A}_n^{-1} + \tilde{A}_{n+1}^{-1}$, and, from (5),

$$\alpha_0 q_n < \tilde{A}_n < 2\alpha_0 q_n. \tag{7}$$

Also, using (2),

$$\tilde{A}_n \geq \alpha_0\gamma^{n-1}, \quad \frac{\tilde{A}_n}{\tilde{A}_{j-1}} \geq \gamma^{n-j}, \quad 0 \leq j \leq n. \tag{8}$$

3 Space of Poincaré Analytic Vector Fields

Let ϕ_t be the flow generated by a continuous vector field X on $\mathbb{T}^2$, $\dot{\theta} = X(\theta)$, $\theta \in \mathbb{T}^2$, and Φ_t its lift to $\mathbb{R}^2$. For some norm $\|\cdot\|$ in $\mathbb{R}^2$, we say that $w_X(\theta_0) = \lim_{t\to\infty} \Phi_t(\theta_0)/\|\Phi_t(\theta_0)\|$ is the *winding ratio* of X for the orbit of $\theta_0 \in \mathbb{T}^2$, if the limit exists and $\lim_{t\to\infty} \|\Phi_t(\theta_0)\| = \infty$. Otherwise, if $\Phi_t(\theta_0)$ is bounded we put $w_X(\theta_0) = 0$. On the other hand, if the limit does not exist or if $\|\Phi_t(\theta_0)\|$ is unbounded but does not tend to infinity, we do not define the winding ratio.

In the two-dimensional case, the set of winding ratios of vector fields X on $\mathbb{T}^2$ is always a subset of $\{w, 0, -w\}$, for some normalised vector w (cf. e.g. [1, 3]). Such a set is called *winding set*, and will be denoted by $w(X)$.

For any vector field X, the existence of an equilibrium is equivalent to $0 \in w(X)$, because it corresponds to a bounded orbit. If the slope of $w \in w(X)$ is an irrational number α, then $w(X) = \{w\}$ (see [3] and references therein). Fixed-point-free flows on $\mathbb{T}^2$ with only one winding ratio are called *Poincaré flows*.

A closed cross section or *transversal* to a vector field on a surface is a simple closed curve, such that the vector field is nowhere tangent to the curve, and intersects all the orbits. A flow on $\mathbb{T}^2$ has a transversal if and only if it is a Poincaré flow. In this case, considering the return map on the transversal, it is possible to reduce the flow to a diffeomorphism of the circle. Therefore, there is an equivalence between these systems and the results are directly related. If w is rational (more precisely its slope), there is a periodic orbit of winding ratio w and every orbit is asymptotic to such. When w is irrational and the flow is C^2, then all the orbits are dense on $\mathbb{T}^2$, and the flow is topologically equivalent to a constant flow with the same winding ratio [6, 14].

We briefly recall the space of vector fields used in [12] that we will also study here for the two-dimensional case.

Let $r > 0$ and the lift of $\mathbb{T}^2$ to a complex neighbourhood of $\mathbb{R}^2$:

$$\mathcal{D}(r) = \left\{\theta \in \mathbb{C}^2 \colon \|\operatorname{Im}\theta\| < \frac{r}{2\pi}\right\},$$

where $\|\cdot\|$ is the ℓ_1-norm on $\mathbb{C}^2$. That is, $\|z\| = |z_1| + |z_2|$, with $|\cdot|$ the usual norm on $\mathbb{C}$. We also denote the inner product as $z \cdot z' = z_1 \bar{z}'_1 + z_2 \bar{z}'_2$.

We will be dealing with analytic vector fields of the form $X = \omega + f$, where $\omega \in \mathbb{R}^2$ and with analytic functions $f \colon \mathcal{D}(r) \to \mathbb{C}^2$, 2π-periodic in each variable θ_i, represented in Fourier series as

$$f(\theta) = \sum_{k \in \mathbb{Z}^2} f_k e^{2\pi i k \cdot \theta}, \quad f_k \in \mathbb{C}^2.$$

More specifically, f belongs to one of the spaces $(\mathcal{A}(r), \|\cdot\|_r)$ and $(\mathcal{A}'(r), \|\cdot\|'_r)$ whose elements are such that the respective norms

$$\|f\|_r = \sum_{k \in \mathbb{Z}^2} \|f_k\| e^{r\|k\|} \quad \text{and} \quad \|f\|'_r = \sum_{k \in \mathbb{Z}^2} (1 + 2\pi\|k\|) \|f_k\| e^{r\|k\|}$$

are finite. Both are Banach spaces.

Some of the above vector fields $X = \omega + f$ generate Poincaré flows on $\mathbb{T}^2$. Note that, if $\|f\|'_r < \|\omega\|$ there is no equilibria. We will be interested in Poincaré flows, especially with "irrational" winding ratio.

4 One-step Renormalisation Operator

In this section fix a vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 \setminus \{0\}$ with $\omega_1 \neq 0$ and slope $\alpha = \omega_2/\omega_1 \geq 1$, and denote $a = [\alpha]$. As in [10], we separate the terms of a vector field X into the following:

Definition 4.1 For $\sigma > 0$ and $\psi \in \mathbb{C}^2$, we define the *far from resonance terms* with respect to ψ to be the ones whose indices are in

$$I^-_\sigma(\psi) = \left\{ k \in \mathbb{Z}^2 \colon |\psi \cdot k| > \sigma\|k\| \right\}.$$

Similarly, the *resonant terms* are $I^+_\sigma(\psi) = \mathbb{Z}^2 \setminus I^-_\sigma(\psi)$. The respective projections $\mathbb{I}^+_\sigma(\psi)$ and $\mathbb{I}^-_\sigma(\psi)$ on the space of vector fields are defined as

$$[\mathbb{I}^\pm_\sigma(\psi)] X(\theta) = \sum_{k \in I^\pm_\sigma(\psi)} X_k e^{2\pi i k \cdot \theta}.$$

Let $\omega' = \{\alpha\}^{-1} T_a^{-1} \omega$, with slope $\alpha' = \{\alpha\}^{-1}$ that corresponds to the shift of the continued fraction expansion of α, so that $\|\omega'\|/\|\omega\| = (1 + \alpha')/(1 + \alpha)$. Denote also by $\hat{\omega}' = \omega'/(\omega' \cdot \omega')$ the normalised vector in the subspace spanned by ω', i.e. $\hat{\omega}' \cdot \omega' = 1$. We denote by $\mathbb{E}(X)$ the spatial average of a vector field, $\mathbb{E}(X) = \int_{\mathbb{T}^2} X(\theta)\, d\theta$, where $d\theta$ is the normalised Lebesgue measure.

Proposition 4.2 *There exists $0 < \kappa < 1$ and $\sigma > 0$ such that, if $0 < \rho' < \rho$ with $\kappa\rho < \rho'$, any $X \in \mathbb{I}^+_\sigma(\omega)\mathcal{A}(\rho')$ has an analytic extension on $T_a\mathcal{D}(\rho)$. The linear map $\mathcal{T}_a(X) = T_a^{-1} X \circ T_a$ from $\mathbb{I}^+_\sigma(\omega)\mathcal{A}(\rho')$ to $\mathcal{A}'(\rho)$ is compact with norm $\|\mathcal{T}_a\| \leq 6\pi a/(\rho' - \kappa\rho) + 3a$.*

The proof of this proposition is contained in Section 7. As it will be seen, it is the existence of $\kappa < 1$ such that $I_\sigma^+(\omega) \subset \{k \in \mathbb{Z}^2 \colon \|T_a^* k\| \leq \kappa \|k\|\}$ that guarantees the above analyticity improvement.

The following theorem (proved in [12] using a "homotopy method") states the existence of a nonlinear change of coordinates U isotopic to the identity, eliminating all the far from resonance terms $I_\sigma^-(\omega') = I_{\sigma/\alpha'}^-(\omega'/\alpha')$ of a vector field X in a neighbourhood of a non-zero constant vector field.

Theorem 4.3 ([12]) *Let $0 < \rho' < \rho$, $\psi \in \mathbb{C}^d \setminus \{0\}$ and $0 < \sigma < \|\psi\|$. If X is in the open ball $\hat{B} \subset \mathcal{A}'(\rho)$ centred at ψ with radius*

$$\hat{\varepsilon} = \frac{\sqrt{6}-2}{12} \sigma \min\left\{\frac{\rho - \rho'}{4\pi}, \frac{3-\sqrt{6}}{6} \frac{\sigma}{\|\psi\|}\right\},$$

there is a diffeomorphism $U \colon \mathcal{D}(\rho') \to \mathcal{D}(\rho)$ isotopic to the identity, satisfying

$$[\mathbb{I}_\sigma^-(\psi)](DU)^{-1} X \circ U = 0, \qquad \textit{and} \qquad U = \mathrm{Id} \textit{ if } [\mathbb{I}_\sigma^-(\psi)]X = 0.$$

The map $\mathcal{U}_\psi \colon \hat{B} \to [\mathbb{I}_\sigma^+(\psi)]\mathcal{A}(\rho')$ given by $X \mapsto (DU)^{-1} X \circ U$ is analytic, and the derivative at ψ is $\mathbb{I}_\sigma^+(\psi)$. Moreover,

$$\|\mathcal{U}_\psi(X) - \psi\|_{\rho'} \leq 2\left(1 + \max\left\{\frac{2}{3}(3-\sqrt{6}), 6(\sqrt{6}+2)\frac{\|\psi\|}{\sigma}\right\}\right) \|X - \psi\|'_\rho.$$

The definition below uses a different order of the linear and nonlinear maps, compared with the approach of [12]. There is no fundamental difference, only a technical adjustment to allow us to evaluate the size of the iterates right after performing $\mathcal{U}$.

Definition 4.4 A map $\mathcal{R}_\omega$ acting on vector fields X is called a *one-step renormalisation* operator if it is of the form

$$\mathcal{R}_\omega(X) = \frac{1}{\hat{\omega}' \cdot \mathbb{E}(X')} X', \quad \text{with} \quad X' = \mathcal{U}_{\omega'/\alpha'} \circ \mathcal{T}_a(X).$$

The nonlinear transformation $\mathcal{U}_{\omega'/\alpha'}$ that eliminates the far from resonance terms $I_\sigma^-(\omega')$, is defined according to Theorem 4.3 with respect to ω'/α'. The linear map $\mathcal{T}_a$ is given by Proposition 4.2.

Since $\mathcal{R}_\omega(\omega) = \omega'$, fixed points of $\mathcal{R}_\omega$ are possible for constant vectors ω such that $\omega' = \omega$. Those correspond to the ones with slope α having a constant continued fraction expansion, i.e. of the form $[a, a, \dots]$, $a \geq 1$. Other periodic points of $\mathcal{R}_\omega$ can be obtained for α with a periodic continued fraction expansion, which, for period p, is of the form $[\overline{a_0, \dots, a_p}]$, $a_0, \dots, a_p \geq 1$. Eventually periodic points appear for slopes of vectors with eventually periodic continued fraction expansion (see [12]).

Proposition 4.5 *Given $\rho' > 0$ and $\sigma > 0$ sufficiently small, we can find $C > 0$ such that the one-step renormalisation operator $\mathcal{R}_\omega$ is a well-defined analytic map from an open ball B of $\mathbb{I}^+_\sigma(\omega)\mathcal{A}(\rho')$ centred at ω with radius $\zeta = C\sigma^2/(a\|\omega'\|)$, to $\mathbb{I}^+_\sigma(\omega')\mathcal{A}(\rho')$. In addition, $\|\mathcal{R}_\omega(\omega+f)-\omega'\|_{\rho'} \leq \|f\|_{\rho'}/\zeta$ and $\|\mathcal{R}_\omega(\omega+f)-\omega'-D\mathcal{R}_\omega(\omega)\,f\|_{\rho'} \leq [\zeta(\zeta-\|f\|_{\rho'})]^{-1}\|f\|^2_{\rho'}$.*

Proof: Suppose $\rho > \rho'$ and $\kappa < 1$ such that $\kappa\rho < \rho'$. We will always make these choices such that $\rho' - \kappa\rho$ is bounded away from zero. Then, by Proposition 4.2, $\mathcal{T}_a(B) \subset B'$, with $B' = \{X \in \mathcal{A}'(\rho) \colon \|X - \omega'/\alpha'\|'_\rho < CC'\sigma^2/\|\omega'\|\}$, where C' is a positive constant. Then, C has to be chosen sufficiently small so that the transformation $\mathcal{U}_{\omega'/\alpha'}$ is valid in a neighbourhood of ω'/α' containing B', according to Theorem 4.3. The same theorem states that $\mathcal{U}_{\omega'/\alpha'}(B') \subset B''$ where $B'' = \{X \in \mathbb{I}^+_{\sigma/\alpha'}(\omega'/\alpha')\mathcal{A}(\rho') \colon \|X - \omega'/\alpha'\|_{\rho'} < CC'C''\sigma/\alpha'\}$, where $C'' > 0$ is an independent constant.

Let F be the complex-valued continuous functional $F(X') = \hat{\omega}' \cdot \mathbb{E}(X')$, with the domain being a neighbourhood $B''' \subset \mathcal{A}(\rho')$ of ω'/α' such that $B'' \subset B'''$ and $F(B''') \subset \{z \in \mathbb{C} \colon |\alpha' z - 1| < 1/2\}$. Writing $f' = X' - \omega'/\alpha'$, it is enough to have $\|f'\|_{\rho'} < \ell(4\sqrt{2}-5)/2 \leq 1/(2\alpha'\|\hat{\omega}'\|)$ with $\omega' = \ell(1,\alpha')$, which is satisfied for a sufficiently small C. Hence $[\hat{\omega}' \cdot \mathbb{E}(X')]^{-1}$ is bounded and analytic in B'' and, for $X' \in B''$, $\|X'/(\hat{\omega}' \cdot \mathbb{E}\,X') - \omega'\|_{\rho'} \leq CC'C''C'''\sigma$, for some independent scalar $C''' > 0$. This, Theorem 4.3 and Proposition 4.2 prove the first part of the claim. The first estimate also follows in the same way.

To show the estimate on the second order remaining of the Taylor expansion of $\mathcal{R}_\omega$ around ω, consider $g \colon z \mapsto \mathcal{R}_\omega(\omega + zf) - \omega'$, $f \in B$. This is an analytic map on an open ball in $\mathbb{C}$ with radius $r = \zeta/\|f\|_{\rho'} > 1$. So, Cauchy's formula gives

$$\begin{aligned} \|g(1) - g(0) - g'(0)\|_{\rho'} &\leq \frac{1}{2\pi} \oint_{|z|=r} \frac{\|g(z)\|_{\rho'}}{|z^2(z-1)|} dz \\ &\leq \frac{1}{r(r-1)} \sup_{\|\xi\|_{\rho'}=\zeta} \|\mathcal{R}_\omega(\omega+\xi) - \omega'\|_{\rho'}. \end{aligned}$$

□

5 Diophantine Numbers and $GL(2,\mathbb{Z})$

Let us review the definition of diophantine numbers and some of its properties, and introduce the set of vectors of the plane with diophantine slope.

Definition 5.1 For an irrational α and $\beta \geq 0$, we say that α is a *diophantine number of order* β if there is a constant $C > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^{2+\beta}}$$

for any $p/q \in \mathbb{Q}$. We denote the set of all the diophantine numbers of order β by $DC(\beta)$. All non-zero vectors ω in $\mathbb{R}^2$ with slope $\alpha \in DC(\beta)$ form the set denoted also by $DC(\beta)$, which is $GL(2,\mathbb{Z})$-invariant. This is equivalent to say that we can find $C > 0$ such that $|\omega \cdot k| > C\|k\|^{-(1+\beta)}$, $k \in \mathbb{Z}^2 \setminus \{0\}$.

If $\alpha \in DC(\beta)$ and using (6), there exists $K > 1$ such that

$$\frac{1}{q_n q_{n+1}} > \left|\alpha - \frac{p_n}{q_n}\right| > \frac{K^{-1}}{q_n^{2+\beta}}, \quad n \geq 0,$$

where p_n, q_n are the convergents of α. This yields, together with (3), (5) and (7), that the diophantine condition can be written in the following ways:

$$\begin{array}{ll} q_{n+1} & < K q_n^{1+\beta}, \\ a_{n+1} & < K q_n^{\beta}, \\ \beta_{n+1}^{-1} & < 2K \beta_n^{-(1+\beta)}, \\ \tilde{A}_{n+1} & < 2K \tilde{A}_n^{1+\beta}, \quad n \geq 0. \end{array} \tag{9}$$

Finally, for $\beta > 0$, one obtains lower and upper bounds for the convergents $F_n \leq q_n < K^{[(1+\beta)^n - 1]/\beta}$, and for the coefficients $a_{n+1} < K^{(1+\beta)^n}$, where F_n is the Fibonacci sequence given by the recurrence formula $F_{n+1} = F_n + F_{n-1}$ (i.e. with $q_n = F_n$ and $a_n = 1$). The sequence F_n has an exponentially growing solution of the form γ^n. All the *constant type numbers*, i.e. the ones inside $DC(0)$, have an exponential bound on the sequence q_n similar to F_n, and $a_n < K$.

The set of the vector fields X close enough to $\mathbb{E}(X)$ with winding ratio $w \in DC(\beta)$ is $\mathcal{R}_\omega$-invariant for any $\omega \in DC(\beta)$. This is easily seen by recalling that if X' is the new vector field after a coordinate transformation isotopic to $T \in GL(2,\mathbb{Z})$, then the winding ratio of X' is $Tw/\|Tw\|$. In particular, $DC(\beta)$ is T_a-invariant, and T_a is the linear map involved in $\mathcal{R}_\omega$, modulo a constant rescaling.

The constant type numbers $DC(0)$ have zero Lebesgue measure, but the *Roth numbers* (see e.g. [4, 9]): $\cap_{\beta>0} DC(\beta)$, have measure one, which implies the full measure of any set $DC(\beta)$, $\beta > 0$. Note that if $\beta_1 < \beta_2$ then $DC(\beta_1) \subset DC(\beta_2)$.

6 The Limit Set of the Renormalisation

Let $\omega_0 \in \mathbb{R}^2 \setminus \{0\}$ with non-zero slope $\alpha_0 = [a_0, a_1, \dots]$. For an initial vector field X_0, define the sequence of its images under renormalisation as $X_{n+1} = \mathcal{R}_{\omega_n}(X_n)$, whenever possible for $n \geq 0$. We denote also $\omega_{n+1} = \mathcal{R}_{\omega_n}(\omega_n)$. Notice that, if $w(X_n) = w(\omega_n)$, then $w(X_{n+1}) = w(\omega_{n+1})$.

The dynamics of ω_n with slope equal to $\alpha_n = [a_n, a_{n+1}, \dots]$, is given by the shift of coefficients, related to the Gauss map. If α_0 is a rational number, then its continued fraction expansion is finite. Therefore, it is not possible to perform the iteration of this procedure for an infinite number of times.

We want to include in the domain of this renormalisation iteration, vectors ω_0 with any slope and vector fields X_0 with also far from resonance terms. So, initially, we allow $\mathcal{R}_{\omega_0}$ to be an "adjustment" procedure, using the change of basis V or S and $\mathcal{U}$. This unique transient step does not influence the qualitative result that follows.

Theorem 6.1 *Let $\rho' > 0$. For any vector $\omega_0 \in \cup_{\beta<1} DC(\beta)$, there exists an open neighbourhood B of ω_0 in $\mathcal{A}(\rho')$ such that: if $X_0 \in B$ and $w(X_0) = w(\omega_0)$, then*

$$\|X_n - \omega_n\|_{\rho'} < K\theta^n, \quad n \geq 0,$$

where $K > 0$ and $0 < \theta < 1$.

The proof of this theorem is included in Section 8. For "irrational winding ratios" not in the condition of Theorem 6.1, the domain of the renormalisation shrinks faster than the rate of convergence of the iteration. Both of these behaviours are determined by the size of the coefficients a_n. If their growth is controlled as for the diophantine case, the renormalisation reduces the perturbation enough to keep the iterates inside the domain.

A corollary of the above theorem relates to the main result in [12] for the two dimensional case. In fact, quadratic irrationals have an eventually periodic continued fraction expansion and are inside $DC(0)$. It is then enough to choose a renormalisation operator $\mathcal{R}$ associated to such ω, with the block of matrices T corresponding to the periodic string. In this way, ω is an eigenvector of T and a fixed point of $\mathcal{R}$. Theorem 6.1 claims the existence of an invariant set contracting towards ω, that includes all vector fields in some neighbourhood with same winding ratio. To show that that is a submanifold, and there is a one-parameter family corresponding to the unstable direction, we refer to the details of the proof (namely the spectral properties of the derivative of $\mathcal{R}$ at ω). The fixed point is therefore hyperbolic.

Another application of Theorem 6.1 is a result for the Poincaré map on the circle, in the same spirit as the one proved originally by Arnol'd [2] in the analytic case: any real analytic circle map with diophantine rotation number α, inside some neighbourhood of the rotation $R_\alpha \colon x \mapsto x + \alpha \mod 1$, is analytically conjugate to R_α, for a norm induced by the one used here for vector fields (cf. Moser [16] for the differentiable case, and Herman [9] and Yoccoz [18] for more general and global results). We have not determined the optimal condition obtained by Yoccoz [19] in the context of the linearisation of germs of analytic diffeomorphisms and the local theorem on analytic conjugacy of circle diffeomorphisms. That condition states that it is necessary and sufficient that the set of winding ratios for which the claim in Theorem 6.1 is expected to be valid strictly contains $\cup_{\beta\geq 0} DC(\beta)$, with slope verifying the Brjuno condition : $\sum_{n\geq 0}\log(q_{n+1})/q_n < +\infty$.

7 The Sets $I_\sigma^+(\omega)$ and I_a^κ

Here it is enough to consider $\omega = (\omega_1, \omega_2)$, $\omega_1, \omega_2 > 0$, with slope $\alpha = \omega_2/\omega_1 > 1$, and $a = [\alpha]$.

Definition 7.1 For $\kappa > 0$ define the set

$$I_a^\kappa = \{k \in \mathbb{Z}^2 \colon \|T_a^* k\| \leq \kappa \|k\|\},$$

and the projection $\mathbb{I}_a^\kappa$ on $\mathcal{A}(r)$, $r > 0$, given by

$$\mathbb{I}_a^\kappa X(\theta) = \sum_{k\in I_a^\kappa} X_k e^{2\pi i k\theta}, \quad X \in \mathcal{A}(r).$$

Let $0 < \sigma < \omega_i$, $i = 1, 2$, and $1/2 < \kappa < 1$. From the respective definitions one can rewrite the form of the cones $I_\sigma^+(\omega)$ and I_a^κ, closed subsets of $\mathbb{Z}^2$. In fact, $I_\sigma^+(\omega)$ is bounded by the lines $k_2 = mk_1$ and $k_2 = lk_1$, and I_a^κ by $k_2 = sk_1$ and $k_2 = rk_1$, where

$$m = -\tfrac{\omega_1-\sigma}{\omega_2+\sigma}, \quad l = -\tfrac{\omega_1+\sigma}{\omega_2-\sigma} \quad \text{and} \quad s = -\tfrac{1-\kappa}{a-1+\kappa}, \quad r = -\tfrac{1+\kappa}{a+1-\kappa}. \tag{10}$$

Note that $l < m < 0$ and $r < s < 0$ if $\kappa > (a + 1)^{-1}$.

Lemma 7.2 *If $\kappa \geq 1 - \|\omega\|^{-1} \min\{a(\omega_1 - \sigma), 2(\omega_2 - \sigma) - a(\omega_1 + \sigma)\}$, then $I_\sigma^+(\omega) \subset I_a^\kappa$. Moreover, considering $\omega = (1, \alpha)$ with $\alpha > 1$ and $\sigma < 1/3$, it is sufficient to have $\kappa \geq 1 - \eta$ where $\eta = (1 - 3\sigma)/3 > 0$.*

Proof: It is enough to check the values of κ for which we have $r \leq l \leq m \leq s$, as given in (10). The result follows from a simple calculation, considering the conditions on σ imposed before. □

7.1 Analyticity Improvement in I_a^κ

In order to prove that Proposition 4.2 holds, we need to show that $X \circ T_a$ is analytic in $\mathcal{D}(\rho)$ and has bounded derivative. From the result of Lemma 7.2 it is enough to notice that, for $X(\theta) = \sum_{I_a^\kappa} f_k e^{2\pi i k\cdot\theta}$,

$$\|X \circ T_a\|_\rho \leq \sum_{k\in I_a^\kappa} \|f_k\| e^{\rho\|T_a^* k\|} \leq \sum_{k\in I_a^\kappa} \|f_k\| e^{(\rho\kappa-\rho')\|k\|} e^{\rho'\|k\|} \leq \|X\|_{\rho'},$$

and

$$\|D(X \circ T_a)\|_\rho \leq 2\pi \sum_{k\in I_a^\kappa} \|T_a^* k\| e^{-\delta\|k\|} \|f_k\| e^{(\rho\kappa+\delta-\rho')\|k\|} e^{\rho'\|k\|} \leq \frac{2\pi}{\delta}\kappa\|X\|_{\rho'},$$

by choosing $0 < \delta < \rho' - \kappa\rho$, and making use of the relation $\sup_{t\geq 0} te^{-\xi t} \leq 1/\xi$ for $\xi > 0$. These bounds imply that $\|\mathcal{T}_a(X)\|'_\rho \leq (1+2\pi\kappa/\delta)\|T_a^{-1}\| \, \|X\|_{\rho'}$. We choose e.g. $\delta = \kappa(\rho' - \kappa\rho)$.

The above for $\mathcal{D}(\rho)$ is true also for $\mathcal{D}(r)$, $r > \rho$ such that $\rho' > r\kappa$. Therefore, $\mathcal{T}_a = \mathcal{I} \circ \mathcal{J}$, where $\mathcal{J} \colon \mathbb{I}_a^\kappa \mathcal{A}(\rho') \to \mathcal{A}'(r)$ is bounded as $\mathcal{T}_a$ above, and the inclusion map $\mathcal{I} \colon \mathcal{A}'(r) \to \mathcal{A}'(\rho)$, $\mathcal{I}(X) = X|_{\mathcal{D}(\rho)}$ is compact.

8 Convergence Result

Here we include the proof of Theorem 6.1, the convergence of the iterates X_n towards the projected space $\mathbb{E}\,\mathcal{A}(\rho') = \mathbb{C}^2$ of constant vector fields. More precisely, we show that the orbit of X_0 approximates exponentially the orbit of ω_0. For all steps of the iteration, we choose the same κ and ρ from Proposition 4.2.

Considering the n-th step, the slope of ω_n is given by $\alpha_n > 1$, and we denote its integer part by $a_n = [\alpha_n]$. While $\alpha_n > 1$ for $n \geq 1$, we can have $\alpha_0 > 0$, but this can be regarded as a transient step (using S) which does not carry any problems into the following analysis. Also, if α_0 is negative, the change of basis V can be used initially, returning to the case of positive slope. Therefore, we will assume here $\alpha_n > 1$, $n \geq 0$. Again, the same can be said about starting within a space of vector fields with more than resonant terms. After an initial transformation, the iterates will always be inside a space restricted to vector fields with no far from resonance terms.

Let $\sigma > 0$ as given in Proposition 4.2. To simplify the notation replace $\mathcal{R}_{\omega_n}$ simply by $\mathcal{R}_n$ and use the projection $\mathbb{P}_n$ of $\mathcal{A}(\rho')$ over ω_n:

$$\mathbb{P}_n f = (\hat{\omega}_n \cdot f)\, \omega_n, \quad f \in \mathcal{A}(\rho'),$$

where $\hat{v} = v/(v \cdot v)$, for every $v \in \mathbb{R}^2 \setminus \{0\}$. According to Proposition 4.5, the domain of each $\mathcal{R}_n$ is the ball $B_n \subset \mathcal{A}(\rho')$ around ω_n with radius ζ_n. The derivative at ω_n is the linear operator given by:

$$D\mathcal{R}_n(\omega_n) = (\mathbb{I} - \mathbb{P}_{n+1}\mathbb{E})\, \mathcal{L}_n,$$

with $\mathcal{L}_n = \alpha_{n+1}\mathbb{I}_\sigma^+(\omega_{n+1}) \circ \mathcal{T}_{a_n}$. This is a compact operator since $\mathbb{I}^+$ is bounded and $\mathcal{T}_{a_n}$ is compact.

The following proposition shows that there is a "super" exponentially shrinking of the non-constant terms, the image of $(\mathbb{I} - \mathbb{E})$, corresponding to the zero eigenvalue of the stable subspace. Recall that $\tilde{A}_n = \prod_{i=0}^n \alpha_i$ and $\alpha_0 q_n < \tilde{A}_n < 2\alpha_0 q_n$, with q_n growing at least exponentially with n. Also, let $\tilde{A}_{-1} = 1$.

Proposition 8.1 *Suppose $\omega_0 = \ell(1, \alpha_0) \in DC(\beta)$, $\beta \geq 0$, $\ell \neq 0$. If $\sigma < \frac{\ell}{2}(1 + 4\gamma^3)^{-1}$, then there exists $c_1, c_2 > 0$ such that*

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j\, (\mathbb{I} - \mathbb{E})\| \leq c_1 e^{-c_2 \Lambda_{j,n}} \|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1}\, (\mathbb{I} - \mathbb{E})\|,$$

for $j = 0, \dots, n$ with $n > 0$, and where

$$\Lambda_{j,n}^{2+\beta} = \frac{\tilde{A}_{n+1}\tilde{A}_n}{\sigma \tilde{A}_{j-1}^{1+\beta}\tilde{A}_{j-1}}.$$

Proof: Consider a vector ω with slope α, $a = [\alpha]$, the matrix T_a and $\lambda = (a + \sqrt{a^2+4})/2 > a$. Let Ω be an orthogonal vector to ω, e.g. $\Omega = (1, -1/\alpha)$. So,

$$T_a^{-1}\omega = \frac{1}{\alpha'}\omega' \quad \text{and} \quad T_a\Omega = -\frac{1}{\alpha}\Omega', \tag{11}$$

where $\Omega' = (1, -1/\alpha')$. The projections of $k \in I_\sigma^+(\omega)$ over ω and Ω are $\mathbb{P}_\omega\, k = (\omega \cdot k)\,\hat{\omega}$ and $(\mathbb{I} - \mathbb{P}_\omega)\, k = (\hat{\Omega} \cdot k)\,\Omega$, respectively.

Introducing the notation $I_n^+ = I_\sigma^+(\omega_n) \setminus \{0\}$, $n \geq 0$, and the corresponding projection $\mathbb{I}_n^+$, define a subset $V_{j,n}^+$ of I_j^+ as

$$V_{j,n}^+ = \{k \in I_j^+ : T_i^* \cdots T_j^*\, k \in I_{i+1}^+, j \leq i \leq n\},$$

for $j = 0, \dots, n$ when $n > 0$. This set includes all the Fourier modes in I_j^+ that will be mapped into I_{n+1}^+ (the only relevant ones since $\mathcal{L}_n$ includes the projection $\mathbb{I}_{n+1}^+$) by the sequence of matrices $T_n^* \cdots T_j^*$. Therefore, for every k in $V_{j,n}^+$, we can write $|\omega_{n+1} \cdot T_n^* \dots T_j^*\, k| \leq \sigma \|T_n^* \dots T_j^*\, k\|$. Notice that $|\omega_{n+1} \cdot T_n^* \dots T_j^*\, k| = |T_j \dots T_n\, \omega_{n+1} \cdot k|$ and, from (11),

$$T_j \dots T_n\, \omega_{n+1} = \left(\prod_{i=j+1}^{n+1} \alpha_i\right) \omega_j.$$

It follows that

$$|\omega_j \cdot k| \prod_{i=j+1}^{n+1} \alpha_i \leq \sigma \|T_n^* \dots T_j^*\, k\|, \quad k \in V_{j,n}^+. \tag{12}$$

A formula for the calculation of $T_n^* \dots T_j^*\, k$, with $k \in V_{j,n}^+$, can be obtained by induction:

$$\begin{aligned} T_n^* \dots T_j^*\, k = \; & (\omega_j \cdot k) \Bigg[\hat{\omega}_{n+1} \prod_{i=j+1}^{n+1} \alpha_i \\ & + \Omega_{n+1} \sum_{m=j}^{n} (\hat{\Omega}_{m+1} \cdot T_m^*\, \hat{\omega}_m) \left(\prod_{i=j+1}^{m} \alpha_i\right) \prod_{i=m+1}^{n} (-\alpha_i)^{-1} \Bigg] \\ & + \Omega_{n+1}\, (\hat{\Omega}_j \cdot k) \prod_{i=j}^{n} (-\alpha_i)^{-1}, \end{aligned} \tag{13}$$

making use of the orthogonality between ω_i and Ω_i for any i, i.e. $\omega_i \cdot \Omega_i = 0$. That yields $T_i^*\, \hat{\omega}_i = (\hat{\Omega}_{i+1} \cdot T_i^*\, \hat{\omega}_i)\, \Omega_{i+1} + (\omega_{i+1} \cdot T_i^*\, \hat{\omega}_i)\, \hat{\omega}_{i+1}$, with $\omega_{i+1} \cdot T_i^*\, \hat{\omega}_i =$

$T_i\,\omega_{i+1}\cdot\hat\omega_i = \alpha_{i+1}$. There is $K>1$ such that the norm of $T_n^*\dots T_j^*\,k$ as given in (13) is bounded by

$$\begin{aligned}\|T_n^*\dots T_j^*\,k\| \le \;& |\omega_j\cdot k|\left[\|\hat\omega_{n+1}\|\prod_{i=j+1}^{n+1}\alpha_i\right.\\ &\left.+2\sum_{m=j}^{n}|\hat\Omega_{m+1}\cdot T_m^*\,\hat\omega_m|\left(\prod_{i=j+1}^{m}\alpha_i\right)\prod_{i=m+1}^{n}\alpha_i^{-1}\right]\\ &+2K\left(\prod_{i=j}^{n}\alpha_i^{-1}\right)\|k\|,\quad k\in V_{j,n}^+.\end{aligned}$$

We now need an upper estimate of the summation in the above inequality. If α is a quadratic irrational (ω and Ω are orthogonal eigenvectors of T), then $\hat\Omega_{m+1}\cdot T_m^*\,\hat\omega_m = 0$ for every m, which is in agreement with [12]. On the other hand, for other values of α, notice that $|\hat\Omega_{m+1}\cdot T_m^*\,\hat\omega_m| < \frac{1}{\ell}(\alpha_{m+1}^{-1}+\alpha_m^{-1}) < \frac{2}{\ell}$. In addition,

$$\begin{aligned}\sum_{m=j}^{n}\left(\prod_{i=j+1}^{m}\alpha_i\right)\prod_{i=m+1}^{n}\alpha_i^{-1} &\le \alpha_{n+1}^{-1}\prod_{i=j+1}^{n+1}\alpha_i\sum_{m=j}^{n}\tilde A_m\tilde A_n^{-1}\\ &\le \gamma^3\alpha_{n+1}^{-1}\prod_{i=j+1}^{n+1}\alpha_i,\end{aligned}$$

because, from the second inequality in (8), we have that

$$\sum_{m=j}^{n}\tilde A_m\tilde A_n^{-1}\le\gamma^3.$$

The hypothesis and the formula $\omega_n = \ell(1,\alpha_n)$, where $\omega_0=\ell(1,\alpha_0)$, imply that $\Gamma_n\sigma<1/2$ with $\Gamma_n = \|\hat\omega_n\| + 4\gamma^3/(\alpha_n\ell)$. Then, using (12) it follows that

$$|\omega_j\cdot k|\le\frac{2K\sigma\|k\|}{1-\Gamma_{n+1}\sigma}\prod_{i=j}^{n}(\alpha_i\alpha_{i+1})^{-1}\tag{14}$$

Given $j\ge 0$, as $\omega_0\in DC(\beta)$, there is a constant $C_0>0$ such that

$$|\omega_j\cdot k| = |\omega_0\cdot T_0^{*-1}\dots T_{j-1}^{*-1}\,k|\prod_{i=1}^{j}\alpha_i \ge \frac{C_0\prod_{i=1}^{j}\alpha_i}{\|T_0^{*-1}\dots T_{j-1}^{*-1}\,k\|^{1+\beta}}.\tag{15}$$

Consider the supremum norm on $\mathbb{Z}^2$, $\|k\|_\infty = \max_{i=1,2}|k_i|$. It is equivalent to $\|\cdot\|$ since $\frac12\|\cdot\|\le\|\cdot\|_\infty\le\|\cdot\|$. Then, $\|T_0^{*-1}\dots T_{j-1}^{*-1}\,k\|_\infty = \|P_{j-1}^{-1}k\|_\infty = \max\{p_{j-1}+p_{j-2}, q_{j-1}+q_{j-2}\}\le b\tilde A_{j-1}\,\|k\|_\infty$, for some $b>0$, implies that

$$\|T_0^{*-1}\dots T_{j-1}^{*-1}\,k\|\le 2b\tilde A_{j-1}\,\|k\|.\tag{16}$$

Now, it follows from inequality (14) together with (15) and (16) that, for all $k \in V_{j,n}^+$,

$$\|k\|^{2+\beta} \geq \frac{C}{\sigma \tilde{A}_{j-1}^{1+\beta}} \left(\prod_{i=1}^{n+1} \alpha_i \right) \prod_{i=j}^{n} \alpha_i = \frac{C \tilde{A}_n \tilde{A}_{n+1}}{\sigma \tilde{A}_{j-1}^{1+\beta} \tilde{A}_{j-1}},$$

with $C = C_0/(2^{3+\beta}\alpha_0 b^{1+\beta} K)$.

Let the operator $\mathbb{V}_{j,n}^+ \colon \mathcal{A}(r) \to \mathcal{A}(\rho')$, $r > \rho'$, be a projection over the indices in $V_{j,n}^+$ together with an analytic inclusion. That is,

$$\|\mathbb{V}_{j,n}^+ f\|_{\rho'} = \sum_{k \in V_{j,n}^+} \|f_k\| e^{r\|k\|} e^{-(r-\rho')\,\|k\|} \leq \|\mathbb{V}_{j,n}^+\| \, \|f\|_r,$$

where $\|\mathbb{V}_{j,n}^+\| \leq \exp\left[-C^{1/(2+\beta)}(r-\rho')\Lambda_{j,n}\right]$. It is possible to include this transformation in the calculation of the sequence

$$\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j \,(\mathbb{I} - \mathbb{E}) = \mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1} \,(\mathbb{I} - \mathbb{E})\, \mathbb{V}_{j,n}^+ \bar{\mathcal{L}}_j,$$

$0 \leq j \leq n$, where $\bar{\mathcal{L}}_j \colon \mathcal{A}(\rho') \to \mathcal{A}(r)$ with $\rho' > \kappa r$, is $\mathcal{L}_j$ followed by an analytic extension. The norm of $\bar{\mathcal{L}}_j$ can be estimated above by $\alpha_{j+1}\alpha_j$ up to the product with a constant. Therefore,

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j \,(\mathbb{I} - \mathbb{E})\| \leq c_1 \exp(-c_2 \Lambda_{j,n}) \|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1} \,(\mathbb{I} - \mathbb{E})\|$$

with c_1, c_2 some positive constants independent either of j or n. $\square$

To complete the description of the eigenspaces of $D\mathcal{R}_n(\omega_n)$, we look at the constant terms, the image of $\mathbb{E}$. That is, we have to solve $G_n(f) = \lambda f$, with

$$G_n(f) = D\mathcal{R}_n(\omega_n)\mathbb{E}\,(f) = (\mathbb{I} - \mathbb{P}_{n+1})\alpha_{n+1} T_n^{-1} \mathbb{E}\,(f).$$

This operator can be re-written in the following way:

$$G_n(f) = \frac{\alpha_{n+1}}{1 + \{\alpha_n\}^2} \begin{bmatrix} -\alpha_n & 1 \\ \{\alpha_n\}\alpha_n & -\{\alpha_n\} \end{bmatrix} f_0.$$

As it has zero determinant its eigenvalues are 0 and $\nu_n = \operatorname{tr}(G_n)$, with $|\nu_n| > 1$ for any $\alpha_n > 0$. The respective eigenvectors are ω_n and Ω_{n+1}, which determine the stable and unstable subspaces corresponding to the constant term.

The constant part of each iterate $X_n = \omega_n + f_n$ can be written as $\mathbb{E}(X_n) = \omega_n + \mathbb{P}_n \circ \mathbb{E}\,(f_n) + (\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(f_n)$. Notice that

$$\begin{aligned} \mathbb{P}_{n+1} \circ \mathbb{E}\,(f_{n+1}) &= \mathbb{P}_{n+1} \circ G_n\,(f_n) + \mathcal{O}_n(\|f_n\|_{\rho'}^2) \\ &= \mathcal{O}_n(\|f_n\|_{\rho'}^2), \end{aligned} \tag{17}$$

as we know from above that the image of G_n is in $(\mathbb{I} - \mathbb{P}_{n+1})\,\mathbb{C}^2$. That is, its size can be estimated by the square of the norm of f_n. The other term, $(\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(f_n)$, can be controlled in a different way, as it is possible to exclude some vector fields in B_n that do not have winding ratio $\omega_n/\|\omega_n\|$.

Lemma 8.2 *Suppose that $X \in B_n$ has winding ratio $\omega_n/\|\omega_n\|$. Then X belongs to the subset*

$$C_n = \left\{X \in B_n \colon \|(\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(X)\| \leq \|(\mathbb{I} - \mathbb{E})X\|_{\rho'}\right\}.$$

Proof: A subset of vector fields $D_n \subset B_n$ that do not cross the line spanned by ω_n can be of the form:

$$D_n = \left\{X \in B_n \colon \|X(\theta) - \mathbb{E}\,X\| < \|(\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(X)\|, \theta \in \mathcal{D}(\rho')\right\}.$$

That is, the non-constant part of $X \in D_n$ at each point θ is less than the distance (given by the norm $\|\cdot\|$) between $\mathbb{E}\,(X)$ and its projection by $\mathbb{P}_n$ over the subspace spanned by ω_n. The slopes of all the vectors $X(\theta)$ are bigger than α_n or always less than α_n, never crossing that value (as for their respective winding ratios). Therefore, since we have that $\|X(\theta) - \mathbb{E}\,X\| \leq \|(\mathbb{I} - \mathbb{E})X\|_{\rho'}$ for every $\theta \in \mathcal{D}(\rho')$, the complementary set of D_n in B_n, contained in C_n, includes all (but not only) vector fields with the same winding ratio as ω_n. □

One can determine the non-constant part of each iterate $X_n = \omega_n + f_n$ by the recurrence formula:

$$(\mathbb{I} - \mathbb{E})\, f_{n+1} = \mathcal{L}_n(\mathbb{I} - \mathbb{E})\, f_n + (\mathbb{I} - \mathbb{E})\, \mathcal{O}_n(\|f_n\|^2_{\rho'}), \quad n \geq 0.$$

By induction, we obtain

$$\begin{aligned} (\mathbb{I} - \mathbb{E})\, f_{n+1} = \;& \mathcal{L}_n \circ \cdots \circ \mathcal{L}_0\, (\mathbb{I} - \mathbb{E})\, f_0 \\ & + \sum_{j=1}^{n} \mathcal{L}_n \circ \cdots \circ \mathcal{L}_j\, (\mathbb{I} - \mathbb{E})\, \mathcal{O}_{j-1}\left(\|f_{j-1}\|^2_{\rho'}\right) \\ & + (\mathbb{I} - \mathbb{E})\, \mathcal{O}_n\left(\|f_n\|^2_{\rho'}\right). \end{aligned} \tag{18}$$

Note that the symbol $\mathcal{O}_n(\|f_n\|^2_{\rho'})$ denotes the quadratic remaining of $\mathcal{R}_n$ as in Proposition 4.5 which we write as F_n^2.

Lemma 8.3 *If $\omega_0 \in DC(\beta)$, $\beta < 1$, there exists $\sigma, C, \tau > 0$ such that, if $\|f_0\|_{\rho'} \leq C^2 < 1$, we have $\|f_n\|_{\rho'} \leq C\tilde{A}_n^{-\tau} < \zeta_n$, $n \geq 1$.*

Proof: We prove this lemma by induction. From the formula $f_n = (\mathbb{I} - \mathbb{E})f_n + \mathbb{P}_n \circ \mathbb{E}\,(f_n) + (\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(f_n)$, and referring to (17) and Lemma 8.2,

$$\|f_n\|_{\rho'} \leq 2\|(\mathbb{I} - \mathbb{E})f_n\|_{\rho'} + F^2_{n-1},$$

where $F^2_{n-1} = \zeta^{-1}_{n-1}(\zeta_{n-1} - \|f_{n-1}\|_{\rho'})^{-1}\|f_{n-1}\|^2_{\rho'}$ according to Proposition 4.5. We write $\zeta_n = c'/(\alpha_n\alpha_{n+1})$ for some constant $c' > 0$. Note that $C/\tilde{A}_n^{\tau} < \frac{1}{2}\zeta_n$ for a good choice of C and τ independent of n, and by using the diophantine conditions.

Notice that $e^{-t} \le (\nu/t)^\nu$ for any $t > 0$ and $\nu > 0$. Therefore, for $\tau > 0$ and $0 \le j \le n$, and the constants $c_1, c_2 > 0$ given in Proposition 8.1:

$$c_1 e^{-c_2 \Lambda_{j,n}} \le c_1 \sigma^\tau \left[\frac{\tau(2+\beta)}{c_2}\right]^{\tau(2+\beta)} \frac{\tilde{A}_{j-1}^{\tau(1+\beta)}}{\tilde{A}_{n+1}^\tau} \le \frac{\tilde{A}_{j-1}^{\tau(1+\beta)}}{7\tilde{A}_{n+1}^\tau}, \tag{19}$$

where we have assumed $\sigma^\tau \le [c_2/(\tau(2+\beta))]^{\tau(2+\beta)}/(7c_1)$, for the last inequality to hold.

If $n = 1$, provided that $C < \frac{1}{2}\zeta_0$ and $2C^2(1 + 1/\zeta_0) < C/\tilde{A}_1^\tau < \frac{1}{2}\zeta_1$, we have

$$\|f_1\|_{\rho'} \le 2C^2 + \frac{C^2}{\zeta_0(\zeta_0 - C)} \le \frac{C}{\tilde{A}_1^\tau} < \frac{1}{2}\zeta_1.$$

Assuming the result is true for $n \ge 1$, it remains to determine a bound of $\|f_{n+1}\|_{\rho'}$. So, we look at each of the terms of the formula (18):

$$\begin{aligned} \|(\mathbb{I} - \mathbb{E})\, f_{n+1}\|_{\rho'} \le \quad & \|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_0 \,(\mathbb{I} - \mathbb{E})\, f_0\|_{\rho'} \\ & + \sum_{j=1}^n \|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j \,(\mathbb{I} - \mathbb{E})\|.F_{j-1}^2 \\ & + F_n^2. \end{aligned} \tag{20}$$

The first two are estimated by the use of Proposition 8.1 through (19). Therefore,

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_0 \,(\mathbb{I} - \mathbb{E})\, f_0\|_{\rho'} \le c_1 C e^{-c_2 \Lambda_{0,n}} \le \frac{C}{7\tilde{A}_{n+1}^\tau}.$$

For the remaining terms we make use of the diophantine conditions in (9) with constants K, $\bar{K}$ and β: $\alpha_{n+1} \le K \tilde{A}_n^\beta$ and $\tilde{A}_{n+1} \le \bar{K} \tilde{A}_n^{1+\beta}$. So,

$$\begin{aligned} F_n^2 \;&\le\; \frac{2C^2}{c'^2} \frac{\alpha_n^2 \alpha_{n+1}^2}{\tilde{A}_n^{2\tau}} \le \frac{2C^2 K^2}{c'^2} \frac{\alpha_n^2 \tilde{A}_n^{2\beta}}{\tilde{A}_n^{2\tau}} \\ &\le\; \frac{2C^2 K^2 \bar{K}^{(2\tau-2-2\beta)/(1+\beta)}}{c'^2} \frac{1}{\tilde{A}_{n+1}^{(2\tau-2-2\beta)/(1+\beta)}} \le \frac{C}{7\tilde{A}_{n+1}^\tau}, \end{aligned}$$

for suitable $C > 0$ and $\tau \ge (2\beta + 2)/(1 - \beta)$. Now,

$$\begin{aligned} \textstyle\sum_{j=1}^n \|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j \,(\mathbb{I} - \mathbb{E})\| F_{j-1}^2 \;&\le\; \frac{2C^2K^2}{7c'^2} \textstyle\sum_{j=1}^n \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+2\beta+2}}{\tilde{A}_{n+1}^\tau \tilde{A}_{j-1}^{2\tau}} \\ &\le\; \frac{2C^2K^2}{7c'^2} \textstyle\sum_{j=1}^n \tilde{A}_{n+1}^{-\tau} \tilde{A}_{j-1}^{-t}, \end{aligned} \tag{21}$$

if $\tau \ge (2\beta + 2 + t)/(1 - \beta)$ for some $t > 0$ and $\beta < 1$. Since $\tilde{A}_{j-1}^{-t}$ can be bounded by $\gamma^{-t(j-1)}$ times a constant, the sum in (21) can be bounded by some $M > 0$. Hence, (21) is estimated by $C/(7\tilde{A}_{n+1}^\tau)$ as long as C is small.

Finally, if C, τ and σ are chosen accordingly to the various conditions described above, then

$$\|f_{n+1}\|_{\rho'} \leq \frac{2(C/7 + C/7 + C/7) + C/7}{\tilde{A}^{\tau}_{n+1}} = \frac{C}{\tilde{A}^{\tau}_{n+1}} < \frac{1}{2}\zeta_{n+1},$$

as required. $\square$

From the lower bound on $\tilde{A}_n$ in (8), $\|f_n\|_{\rho'}$ decreases with n at least geometrically like $\gamma^{-\tau n}$. That completes the proof of Theorem 6.1.

Acknowledgements

I am indebted to Professor R. S. MacKay for the orientation and multiple suggestions given. I also wish to acknowledge Kostya Khanin and Nuno Romão for useful discussions. The author is supported by Fundação para a Ciência e a Tecnologia, under the research grant BD/11230/97.

References

[1] D. V. Anosov. On the behaviour in the Euclidean or Lobachevsky plane of trajectories that cover trajectories of flows on closed surfaces I. *Math. USSR Izvestiya*, 30:15–38, 1988.

[2] V. I. Arnol'd. Small denominators I, mappings of the circumference onto itself. *Transl. AMS 2nd Series*, 46:213–284, 1961.

[3] C. Baesens, J. Guckenheimer, S. Kim, and R. S. MacKay. Three coupled oscillators: mode-locking, global bifurcations and toroidal chaos. *Physica D*, 49:387–475, 1991.

[4] J. W. S. Cassels. *An introduction to diophantine approximations.* CUP, 1957.

[5] C. Chandre and P. Moussa. Scaling law for the critical function of an approximate renormalization. *Nonlinearity*, 14:803–816, 2001.

[6] A. Denjoy. Sur les courbes définies par les équations différentielles à la surface du tore (On the curves defined by the differential equations on the torus surface). *J. Math.*, 17:333–375, 1932.

[7] M. J. Feigenbaum. Quantitative universality for a class of non-linear transformations. *J. Stat. Phys.*, 19:25–52, 1978.

[8] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers.* Oxford Science Publications, 5th edition, 1990.

[9] M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations (On the differentiable conjugacy of the diffeomorphims of the circle to rotations). *Publ. Math. Inst. Hautes Étud. Sci.*, 49:5–233, 1979.

[10] H. Koch. A renormalization group for Hamiltonians, with applications to KAM tori. *Erg. Theor. Dyn. Syst.*, 19:475–521, 1999.

[11] S. Lang. *Introduction to diophantine approximations.* Springer-Verlag, 2nd edition, 1995.

[12] J. Lopes Dias. Renormalization of flows on the multidimensional torus close to a KT frequency vector. *Nonlinearity*, 15, 2002. to appear.

[13] R. S. MacKay. Exact results for an approximate renormalisation scheme and some predictions for the breakup of invariant tori. *Physica D*, 33:240–265, 1988.

[14] R. S. MacKay. A simple proof of Denjoy's theorem. *Math. Proc. Cambridge Phil. Soc.*, 103:299–303, 1988.

[15] R. S. MacKay. *Renormalisation in area-preserving maps.* World Scientific Publishing Co. Inc., River Edge, NJ, 1993.

[16] J. Moser. A rapidly convergent iteration method and non-linear partial differential equations. II. *Ann. scient. Scuola Norm. Sup. di Pisa, série III*, 20:499– 535, 1966.

[17] J. Stark. Smooth conjugacy and renormalisation for diffeomorphisms of the circle. *Nonlinearity*, 1:541–575, 1988.

[18] J.-C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne (Smooth conjugacy of diffeomorphisms of the circle whose rotation number is of diophantine type). *Ann. Sci. Éc. Norm. Super.*, 17:333–359, 1984.

[19] J.-C. Yoccoz. Petits diviseurs en dimension 1 (Small divisors in dimension one). *Astérisque*, 231, 1995.