REDUCIBILITY OF QUASI-PERIODICALLY FORCED CIRCLE FLOWS

SAŠA KOCIĆ AND JOÃO LOPES DIAS

Abstract. We develop a renormalization group approach to the problem of reducibility of quasi-periodically forced circle flows. We apply the method to prove a reducibility theorem for such flows.

1. Introduction

In this paper, we study the dynamics of quasi-periodically time-dependent ordinary differential equations on the circle $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. These equations correspond to skew-product flows generated by vector fields on $\mathbb{T}^d \times \mathbb{T}^1$ whose dynamics are given by

$$\begin{aligned} \dot{x} &= \omega, \\ \dot{y} &= f(x, y), \end{aligned} \tag{1.1}$$

where $(x, y) \in \mathbb{T}^d \times \mathbb{T}^1$, $\omega \in \mathbb{R}^d$ and $f : \mathbb{T}^d \times \mathbb{T}^1 \to \mathbb{R}$ is real analytic. An important problem in the dynamics of ordinary differential equations is to establish conditions under which one can analytically conjugate the flow ϕ^t generated by (1.1) to the linear flow of a constant vector field. If that is possible, we say that f is analytically or C^ω-reducible.

An important conjugacy invariant is the rotation number of f, i.e., its time average along the orbit,

$$\operatorname{rot} f = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f \circ \phi^s(x, y)\, ds, \tag{1.2}$$

for any $(x, y) \in \mathbb{T}^d \times \mathbb{T}^1$ (see Section 2.3).

If f depends only on y (or $\omega = 0$), the dynamical system given by (1.1) is integrable, since the second equation then yields an autonomous vector field on the circle. In that case, zeros of f correspond to the fixed points of the dynamics and all orbits converge to them. If f has no zeros, then all orbits are periodically winding around the circle and the flow is uniquely ergodic for some absolutely continuous invariant measure μ. The reducibility conjugacy can then be constructed using the solution ϕ_2^t of the second equation in (1.1), as $(x, y) \mapsto (x, \phi_2^T y)$,

Date: July 11, 2016.

where T is the least period of the motion given by the frequency of the system $T^{-1} = \operatorname{rot} f = \int_{\mathbb{T}^1} f\, d\mu$.

We are interested in the general case in which f depends on both x and y, i.e., non-autonomous circle flows. When $d = 1$, the system (1.1) corresponds to a periodic perturbation of the circle flow, i.e., to a flow on $\mathbb{T}^2$. Herman's theory [8] (further developed by Yoccoz [21]) implies that if $\operatorname{rot} f/\omega$ satisfies Yoccoz's $\mathcal{H}$ arithmetical condition [21, 22], then the system is C^ω-reducible.

Similar conclusions arise in the case when $\omega \in \mathbb{Q}^d \setminus \{0\}$, $d > 1$. Without loss of generality, up to a time rescaling, we can take $\omega \in \mathbb{Z}^d$. By a linear change of the basis for the torus $\mathbb{T}^d$, we can reduce our initial system to $\dot{x} = (1, 0, \dots, 0)$, $\dot{y} = f(x, y)$. By writing $x = (t, \lambda)$, we obtain the systems: $\dot{t} = 1$, $\dot{\lambda} = 0$, $\dot{y} = F_\lambda(t, y) = f((t, \lambda), y)$. For each $\lambda \in \mathbb{T}^{d-1}$, this corresponds again to a vector field on $\mathbb{T}^2$.

We will now restrict our considerations to $\omega \in \mathbb{R}^d \setminus \mathbb{Q}^d$, $d > 1$. In fact, we will focus only on incommensurate frequency vectors ω, i.e., vectors whose components are rationally independent. For incommensurate ω, the main difficulty in the analysis is related to the existence of small divisors. In this case, there are already some results obtained by KAM (Kolmogorov-Arnol'd-Moser)-type methods [1, 2, 16, 19]. In this paper, we develop a different approach to the problem, based on a renormalization method. This renormalization approach is also different than that exploited in [5, 6, 7], which is based on resummation of a perturbation series in analogy to quantum field theory. While in KAM theory one typically encounters small divisors in a finite number of narrow regions surrounding some resonant planes, in the problem at hand, we encounter and develop an approach to deal with an infinite number of resonant planes (see Remark 1.4). The approach developed here should, therefore, also be useful for the construction of quasiperiodic solutions, i.e., invariant tori, for PDEs, where one necessarily has to deal with an infinite number of resonant planes.

On a space of vector fields $X = (\omega, f)$ of the form (1.1), we define a renormalization operator $\mathcal{R}$ (see Section 3) as

$$\mathcal{R}(X) = \eta^{-1} \mathcal{T}^* \mathcal{U}_X^*(X), \tag{1.3}$$

where $\eta \in (0, 1)$ is the time rescaling parameter, $\mathcal{U}_X$ is a change of variables chosen such that $\mathcal{U}_X^*(X)$, i.e., the pullback of X under $\mathcal{U}_X$, is in an appropriate normal form (see Section 3.2), and $\mathcal{T}$ is a scaling $(x, y) \mapsto (Tx, y)$, defined by a matrix $T \in \mathrm{SL}(d, \mathbb{R})$. The transformation $\mathcal{R}$ will be constructed such that it preserves the form of the vector field $X = (\omega, f)$ and we will define the induced map $\mathfrak{R} : f \mapsto \mathfrak{R}(f)$ by $\mathcal{R}(\omega, f) = (\omega, \mathfrak{R}(f))$. Note that f is a non-autonomous vector field on a circle and $\mathfrak{R}$ defines a renormalization operator on a Banach space of these vector fields.

We will consider vector fields close to $Y = (\omega, \theta)$, with $\omega \in \mathrm{BC}$ and $\theta \in \mathrm{DC}_\omega$. We denote by BC the set of Brjuno vectors $\omega \in \mathbb{R}^d$, i.e., vectors that satisfy [3, 4, 20]

$$\sum_{n=1}^{\infty} 2^{-n} \ln(1/\Omega_n) < \infty\,, \qquad \Omega_n = \min_{\nu \in \mathcal{V},\, 0<|\nu|<2^n} |\omega \cdot \nu|, \tag{1.4}$$

where $\mathcal{V} = \mathbb{Z}^d$. Here, $|\cdot|$ denotes the ℓ^1 norm of a vector in $\mathbb{R}^d$, and dot denotes the usual scalar product of vectors in $\mathbb{R}^d$. Given $\omega \in \mathbb{R}^d$, $\tau, \varkappa \geq 0$, and $C > 0$, we define $\mathrm{DC}_\omega(C, \tau, \varkappa)$ to be the set of all $\theta \in \mathbb{R} \setminus \{0\}$ such that

$$|\omega \cdot \nu + k\theta| > \frac{C}{|\nu|^{d+\tau}|k|^{\varkappa}}, \quad \text{for all} \quad k \in \mathbb{Z}\backslash\{0\}, \quad \nu \in \mathcal{V} \setminus \{0\}. \tag{1.5}$$

We further define

$$\mathrm{DC}_\omega(\tau, \varkappa) = \bigcup_{C>0} \mathrm{DC}_\omega(C, \tau, \varkappa) \quad \text{and} \quad \mathrm{DC}_\omega = \bigcup_{\tau, \varkappa \geq 0} \mathrm{DC}_\omega(\tau, \varkappa).$$

Since we will perform scaling with matrices $T \in SL(d, \mathbb{R})$, we will consider functions with periodicity of a simple lattice $\mathcal{Z}$ in $\mathcal{R}^d$ that is more general that $2\pi\mathbb{Z}^d$. Functions that are invariant under $\mathcal{Z}$-translations can be identified with functions on $\mathbb{T}^d := \mathbb{R}^d/\mathcal{Z}$ or, equivalently with quasiperiodic functions on $\mathbb{R}^d$ with frequency module in the dual lattice $\mathcal{V}$ (the set of points $v \in \mathbb{R}^d$ satisfying $e^{iv\cdot z} = 1$, for all $z \in \mathcal{Z}$). For convenience, we will perform a linear change of coordinates in $\mathbb{R}^d$ such that $\omega = (1, 0, \ldots, 0)$. The lattice obtained from $2\pi\mathbb{Z}^d$ under this change of coordinates in $\mathbb{R}^d$ will be denoted by $\mathcal{Z}_0$ and its dual lattice by $\mathcal{V}_0$.

We consider vector fields X of the form $X = (\omega, f)$ that are close to $Y = (\omega, \theta)$, with f analytic on a complex neighborhood of $D_{\rho,r}$ of $\mathbb{T}^d \times \mathbb{T}^1$ characterized by $|\operatorname{Im} x_i| < \rho$ and $|\operatorname{Im} y| < r$. In the following, we will refer to these vector fields as vector fields of the form (ω, f). In Section 2.2, we will introduce the spaces of vector fields of the form (ω, f) with f analytic on $D_{\rho,r}$, with frequency module in $\mathcal{V}$, and the corresponding Banach spaces $\mathcal{A}_{\rho,r}(\mathcal{V})$ of functions f. If $r = \rho$, we will denote these spaces and the corresponding domains simply by $\mathcal{A}_\rho(\mathcal{V})$ and D_ρ, respectively.

Let $\mathbb{E}$ be a projection operator onto the subspace of constant vector fields (either on the circle or on $\mathbb{T}^d \times \mathbb{T}^1$), given by the averaging of the function over $\mathbb{T}^d \times \mathbb{T}^1$ (see Section 2.2).

The main results of this paper can be summarized in the following theorem.

Theorem 1.1. *Let $\varrho, r > 0$ and let $\omega \in \mathrm{BC}$ and $\theta \in \mathrm{DC}_\omega$. There exist a sequence of matrices $T_n \in \mathrm{SL}(d, \mathbb{R})$, a sequence of time rescaling parameters $\eta_n \in (0, 1)$, and a corresponding sequence of renormalization operators $\mathcal{R}_n$, $n \in \mathbb{N}$, of the form* (1.3)*, such that the corresponding operators $\mathfrak{R}_n$ are analytic from an open neighborhood $\mathcal{D}_{n-1}$ of θ_{n-1}, where $\theta_n = \eta_n^{-1}\theta_{n-1}$, $\theta_0 = \theta$, in $\mathcal{A}_{\rho_{n-1}, r_{n-1}}(\mathcal{V}_{n-1})$, to $\mathcal{A}_{\rho_n, r_n}(\mathcal{V}_n)$, where $\mathcal{V}_n = T_n\mathcal{V}_{n-1}$. The set $\mathcal{W}$ of infinitely renormalizable vector fields f_0 in $\mathcal{D}_0$, characterized by the property that $f_n = \mathfrak{R}_n(f_{n-1})$ belongs to $\mathcal{D}_n$, for every $n \in \mathbb{N}$, is the graph of an analytic function $W : (\mathbb{I} - \mathbb{E})\mathcal{D}_0 \to \mathbb{E}\mathcal{D}_0$ that satisfies $W(0) = \theta_0$ and $DW(0) = 0$. In particular, if $f \in \mathcal{D}_0$ and $\operatorname{rot} f = \theta$, then $f \in \mathcal{W}$. If $\rho > \varrho + \delta$, $\rho > r + \delta$ and $\delta > 0$, then every vector field $X = (\omega, f)$ with $f \in \mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$ is analytically reducible via an analytic conjugacy of the form $\Gamma_X = \mathrm{id} + (0, \psi_X)$, with $\psi_X \in \mathcal{A}_\delta(\mathcal{V}_0)$, that conjugates the flow of X and the flow of $Y = (\omega, \theta)$.*

An immediate corollary of Theorem 1.1 is the following.

Corollary 1.2. *For every $\omega \in \mathrm{BC}$ and $\theta \in \mathrm{DC}_\omega$, there is an open ball B centered at θ in $\mathcal{A}_\rho(\mathcal{V}_0)$, such that every vector field $X = (\omega, f)$, with $f \in B$ and $\operatorname{rot} f = \theta$, is analytically reducible to $Y = (\omega, \theta)$.*

Remark 1.3. Notice that the renormalization operators $\mathcal{R}_n$ are well-defined on $\{\omega\} \times \mathcal{D}_n$, which are open in the space of vector fields of the form (ω, f), and that $\{\omega\} \times \mathcal{W}$ is the stable manifold for this sequence of renormalization operators.

Remark 1.4. The renormalization approach developed here is similar to, but technically more involved than, the renormalization approach to the construction of invariant tori for Hamiltonian and other vector fields [10, 11, 12, 14, 17, 18], reducibility of skew-product flows on $\mathbb{T}^d \times \mathrm{SL}(2, \mathbb{R})$ [15] and construction of lower-dimensional tori for Hamiltonian flows [13]. The small divisors encountered in these problems are produced by frequencies $\nu \in \mathbb{Z}^d$ that lie in the "resonant" regions, outside of certain "non-resonant" cones, surrounding some resonant planes, perpendicular to ω. In the case of maximal-dimensional KAM tori, the small divisors are given by $|\omega \cdot \nu|$ and, thus, there is only one such a plane [10, 11, 12, 14]. The renormalization transformations (see Section 3) then eliminate the non-resonant modes of a vector field and transform some of the remaining resonant modes into non-resonant. In the case of reducibility of skew-product flows on $\mathbb{T}^d \times \mathrm{SL}(2, \mathbb{R})$, there is an additional resonant plane surrounded by frequencies corresponding to small divisors $|\omega \cdot \nu - 2\rho|$, where $\pm i\rho$ are the eigenvalues of a matrix in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ [15]. In the case of lower-dimensional tori [13], there are finitely many resonant planes corresponding to small divisors $|\omega \cdot \nu + \Omega \cdot V|$, where $\Omega \in \mathbb{R}^D$ is vector of normal frequencies and $0 < |V| \leq 2$. In the problem at hand, we encounter and develop a renormalization approach to deal with an infinite number of resonant planes, corresponding to small divisors of the form $|\omega \cdot \nu + k\theta|$, for $k \in \mathbb{Z}$.

The paper is organized as follows. In Section 2, we introduce the spaces of vector fields that we consider. In Section 3, we construct the renormalization scheme and prove the convergence of vector fields of the form (1.1), on the stable manifold of the renormalization operator, towards the constant vector field. In Section 4, we construct analytic conjugacy between the flows of a vector field $X = (\omega, \theta)$ on the stable manifold and a constant vector field $Y = (\omega, \theta)$, and prove Theorem 1.1.

2. Preliminaries

2.1. **Skew-product vector fields and changes of coordinates.** Recall that we are interested in skew-product vector fields on $\mathbb{T}^d \times \mathbb{T}^1$, of the form

$$X(x, y) = (\omega, f(x, y)), \tag{2.1}$$

with $\omega \in \mathbb{R}^d$ and $f \colon \mathbb{T}^d \times \mathbb{T}^1 \to \mathbb{R}$. We will refer to $\mathbb{T}^d$ as the base and to $\mathbb{T}^1$ as the fiber. The dynamics generated by X on the base is trivially given by $x \mapsto x + \omega\, t \mod \mathcal{Z}$.

We will consider real analytic diffeomorphisms $H \in \mathrm{Diff}^\omega(\mathbb{T}^d \times \mathbb{T}^1)$ which preserve the space of skew-product vector fields and are of the type

$$H(x, y) = (x, y + h(x, y)), \tag{2.2}$$

where $(x, y) \in \mathbb{T}^d \times \mathbb{T}^1$ and $h \in C^\omega(\mathbb{T}^d \times \mathbb{T}^1, \mathbb{T}^1)$. We call them skew-product diffeomorphisms.

The action of H on $X = (\omega, f)$ is given by the pull-back

$$H^* X = (DH)^{-1}\, X \circ H.$$

As the form of the vector field is preserved, we abuse the notation in order to write the pull-back as acting on the fiber component of the vector field

$$H^* f = (1 + \partial_y h)^{-1}(-\omega \cdot \partial_x h + f \circ H). \tag{2.3}$$

The flow ϕ'^t generated by H^*X is related to the flow ϕ^t of X by

$$\phi'^t = H^{-1} \circ \phi^t \circ H. \tag{2.4}$$

The vector fields and skew-product diffeomorphisms considered are real-analytic and, thus, can be extended to a complex domain.

2.2. **Spaces and norms.** We will use $\|\cdot\|$ and $|\cdot|$ to denote the ℓ^∞ and ℓ^1 norms, respectively, of a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Let $\rho, r > 0$ and let

$$D_{\rho,r} = \left\{(x,y) \in \mathbb{C}^d \times \mathbb{C} \colon \|\operatorname{Im} x\| < \rho, |\operatorname{Im} y| < r\right\}. \tag{2.5}$$

In this paper, we consider functions with periodicity of $\mathcal{Z} \times 2\pi\mathbb{Z}$, where $\mathcal{Z}$ is a lattice $\mathcal{Z} \subset \mathbb{R}^d$. Recall that for a lattice $\mathcal{Z} \subset \mathbb{R}^d$, the dual lattice is defined as

$$\mathcal{V} = \{v \in \mathbb{R}^d \colon (\exists z \in \mathcal{Z})\, e^{iz\cdot v} = 1\}. \tag{2.6}$$

We will denote by $\mathcal{N}$, the lattice $\mathbb{Z}$ which is dual to $2\pi\mathbb{Z}$.

The norm of a function f, analytic on $D_{\rho,r}$, that can be expanded as

$$f(x,y) = \sum_{v\in\mathcal{V}, k\in\mathcal{N}} f_{v,k}\, e^{ix\cdot v + iyk}, \tag{2.7}$$

is given by

$$\|f\|_{\rho,r} = \sum_{v\in\mathcal{V}, k\in\mathcal{N}} |f_{v,k}|\, e^{\rho|v| + r|k|}. \tag{2.8}$$

Given any $K \in \mathbb{N}$, we denote by $\mathbb{I}_K f$ the truncation of f corresponding to the modes with $|k| \le K$, i.e,

$$\mathbb{I}_K f(x,y) = \sum_{v\in\mathcal{V}, k\in\mathcal{N}, |k|\le K} f_{v,k}\, e^{ix\cdot v + iyk}.$$

We denote by $\mathbb{I}$ the identity operator acting as $\mathbb{I}f = f$, and by $\mathbb{E}$ the average of f, given by the action

$$\mathbb{E}f = \int_{\mathbb{T}^1}\int_{\mathbb{T}^d} f(x,y)\, dx\, dy = f_{0,0}.$$

The Banach space of functions f, analytic on $D_{\rho,r}$, for which the norm $\|f\|_{\rho,r}$ is finite will be denoted by $\mathcal{A}_{\rho,r}(\mathcal{V})$. Similarly, $\mathcal{A}'_{\rho,r}(\mathcal{V})$ is the Banach spaces of functions f, analytic on $D_{\rho,r}$, for which the norm

$$\|f\|'_{\rho,r} = \|f\|_{\rho,r} + \sum_{v\in\mathcal{V}, k\in\mathcal{N}} (\|v\| + |k|)\, |f_{v,k}|\, \mathrm{e}^{\rho|v| + r|k|},$$

is finite. Whenever there is no ambiguity, we avoid writing $\mathcal{V}$ explicitly.

We present several properties of the above norms, which will be used throughout the paper without an explicit reference to them.

Lemma 2.1. *Let $f, g \in \mathcal{A}'_{\rho,r}$, $r' < r$, $K \in \mathbb{N}$ and $\delta > 0$. Let also $U(x,y) = (x, y + u(x,y))$ be a skew-product diffeomorphism satisfying $\|u\|_{\rho,r'} < (r - r')/2$. Then,*

- $\sup_{x\in D_{\rho,r}} |f(x)| \le \|f\|_{\rho,r} \le \|f\|'_{\rho,r} \le (1+2\delta^{-1})\,\|f\|_{\rho+\delta,r+\delta}$,
- $\|(\mathbb{I}-\mathbb{I}_K)f\|_{\rho,r'} \le e^{-K(r-r')}\|f\|_{\rho,r}$,
- $\|f\,g\|_{\rho,r} \le \|f\|_{\rho,r}\|g\|_{\rho,r}$,
- $\|f\,g\|'_{\rho,r} \le \|f\|'_{\rho,r}\|g\|'_{\rho,r}$.
- $\|f\circ U\|_{\rho,r'} \le \|f\|_{\rho,r}$,
- $\|f\circ U - f\|_{\rho,r'} \le 2(r-r')^{-1}\|f\|_{\rho,r}\|u\|_{\rho,r'}$,

Proof. The proof of these inequalities is straightforward and will be omitted. In the proof of the last inequality, one uses Cauchy's estimate for the derivative. □

2.3. **Rotation number.** Define the (fibered) rotation number of f at $(x,y)\in\mathbb{R}^d\times\mathbb{R}$ as

$$\operatorname{rot} f(x,y) = \lim_{t\to+\infty} \frac{\widetilde{\phi}_2^t(x,y)-y}{t} \qquad \bmod 2\pi,$$

where $\widetilde{\phi}_2^t = \pi_2\widetilde{\phi}^t$ stands for the last component of a lift $\widetilde{\phi}^t$ of the flow ϕ^t to the universal cover $\mathbb{R}^{d+1}$. Some versions of the following claim can be found in the literature. We include here a version relevant for our systems.

Proposition 2.2. *If $f\in C^0(\mathbb{T}^d\times\mathbb{T}^1,\mathbb{R})$ and $\omega\in\mathbb{R}^d$ is incommensurate with respect to $\mathcal{Z}$, then $\operatorname{rot} f$ exists and it is constant everywhere on $\mathbb{R}^d\times\mathbb{R}^1$.*

Proof. Notice that, for every $x\in\mathbb{T}^d$ and every $t\in\mathbb{R}$, the map $y\mapsto \phi_2^t(x,y)$ is an orientation-preserving diffeomorphism of the circle satisfying $\widetilde{\phi}_2^t(x,y+2\pi)=\widetilde{\phi}_2^t(x,y)+2\pi$. So, if $y<y'<y+2\pi$, one gets $|\widetilde{\phi}_2^t(x,y')-\widetilde{\phi}_2^t(x,y)|<2\pi$. Assume that the rotation number exists for some (x,y). Hence,

$$\left|\frac{\widetilde{\phi}_2^t(x,y')-y'}{t}-\frac{\widetilde{\phi}_2^t(x,y)-y}{t}\right| \le \left|\frac{\widetilde{\phi}_2^t(x,y')-\widetilde{\phi}_2^t(x,y)}{t}\right|+\left|\frac{y'-y}{t}\right| < \frac{4\pi}{t},$$

for all $t>0$. Taking the limit $t\to+\infty$, one obtains that $\operatorname{rot} f(x,y)=\operatorname{rot} f(x)$ does not depend on y. Taking, e.g., $y=0$, it remains to show that $\operatorname{rot} f(x)$ exists for all x and does not depend on x.

Let $A_t(x)=\widetilde{\phi}^t(x,0)$ and $A_t^q(x)=\widetilde{\phi}^t(x,q)$. Therefore, $A_t^q(x+p)=A_t(x)+(p,q)$ for $(p,q)\in\mathcal{Z}\times(2\pi\mathbb{Z})$. Moreover, define $a_t(x)=\pi_2A_t(x)$ and $a_t^q(x)=\pi_2A_t^q(x)$. We want to show that $\lim_{t\to+\infty}a_t(x)/t$ exists and is independent of x.

We begin with some estimates. For $s, s' > 0$ and $(p,q) \in \mathcal{Z} \times (2\pi\mathbb{Z})$ satisfying $\|A_s(x) - (p,q)\| < 1$, we have

$$\begin{aligned}\frac{a_t(x)}{t} =& -\frac{1}{s't}\int_0^t [a_s \circ A_{s'}(x) - a_s(x) - \frac{s'}{t} a_t(x)]\, ds \\ &+ \frac{1}{s't}\int_0^t [a_s \circ A_{s'}(x) - a^q_{s'}(x+\omega s + p) - a_s(x) + q]\, ds \\ &+ \frac{1}{s't}\int_0^t a_{s'}(x+\omega s)\, ds.\end{aligned}$$

We can easily bound the absolute values of the first two terms. For the first one, by noticing that $\int_0^t (a_{s+s'} - a_s) ds = \int_t^{t+s'} a_s ds - \int_0^{s'} a_s ds = \int_0^{s'} (a_{s+t} - a_s) ds$, we get

$$\int_0^t [a_s \circ A_{s'}(x) - a_s(x) - \frac{s'}{t} a_t(x)]\, ds = \int_0^{s'} [a_s \circ A_t(x) - a_s(x) - a_t(x)]\, ds,$$

whose absolute value is bounded from above by s' times

$$\begin{aligned}M_{s'}(x) =& \max_{0 \le s \le s'} |a_s \circ A_t(x) - a_t(x) - a_s(x)| \\ \le& 2 \max_{0 \le s \le s', (x',y) \in \mathbb{T}^{d+1}} |\widetilde{\phi}^s_2(x',y) - y|.\end{aligned}$$

The second term is bounded by

$$\begin{aligned}&\int_0^t |a_s \circ A_{s'}(x) - a^q_{s'}(x+\omega s+p) - a_s(x) + q|\ ds \\ \le& \int_0^t |a_s \circ A_{s'}(x) - a^q_{s'}(x+\omega s+p)|\ ds + \int_0^t |a_s(x) - q|\ ds < 4\pi t.\end{aligned}$$

So,

$$\left| \frac{a_t(x)}{t} - \frac{1}{t}\int_0^t \frac{a_{s'}(x+\omega s)}{s'}\, ds \right| \le \frac{M_{s'}(x)}{t} + \frac{4\pi}{s'}.$$

Taking the limit $t \to +\infty$, the first term on the right hand side approaches zero. Using Birkhoff's ergodic theorem, since the base flow $x \mapsto x + \omega t \mod \mathcal{Z}$ is uniquely ergodic with respect to the Lebesgue measure dm, we obtain

$$-\frac{4\pi}{s'} + \int_{\mathbb{T}^d} \frac{a_{s'}}{s'}\, dm \le \liminf_{t\to+\infty} \frac{a_t(x)}{t} \le \limsup_{t\to+\infty} \frac{a_t(x)}{t} \le \frac{4\pi}{s'} + \int_{\mathbb{T}^d} \frac{a_{s'}}{s'}\, dm.$$

Finally, taking $s' \to +\infty$, this shows that the rotation number $\operatorname{rot} f(x)$ exists and that $\operatorname{rot} f(x) = \int_{\mathbb{T}^d} \operatorname{rot} f\, dm$ does not depend on x. $\square$

We will use the following properties of the rotation number.

Lemma 2.3. *Let $f, h \in C^0(\mathbb{T}^d \times \mathbb{T}^1, \mathbb{R})$. Then, we have*

- $|\mathbb{E}f - \operatorname{rot} f| \le \|f - \mathbb{E}f\|_{C^0}$.
- $\operatorname{rot}(\tau H^* f) = \tau \operatorname{rot} f$,

for any $\tau \in \mathbb{R}$ and a skew-product diffeomorphism $H(x,y) = (x, y + h(x,y))$.

Proof. Let $\theta = \operatorname{rot} f$. As $\lim_{t\to+\infty} \frac{1}{t}\int_0^t [f \circ \phi^s(x,y) - \theta]\, ds = 0$, we immediately have
$$|\mathbb{E}f - \theta| \le \max |\mathbb{E}f - f|.$$
This proves the first part of the claim.

If we denote by ϕ'^t the flow of $\tau H^* f$, by identity (2.4), we have
$$\begin{aligned}
\operatorname{rot} f &= \lim_{t\to+\infty} \frac{\pi_2 H \circ \widetilde{\phi}'^{t/\tau} \circ H^{-1}(0)}{t} \\
&= \tau^{-1} \lim_{t\to+\infty} \frac{\widetilde{\phi}_2'^{t/\tau} \circ H^{-1}(0) + h \circ \widetilde{\phi}_2'^{t/\tau} \circ H^{-1}(0)}{t/\tau} \\
&= \tau^{-1} \operatorname{rot}(\tau H^* f).
\end{aligned}$$
This completes the proof. □

2.4. **Arithmetics.**

Lemma 2.4. *If $\tau, \varkappa > 0$, there is $\kappa > 0$ such that for any $C > 0$ the Lebesgue measure of the complement of $\mathrm{DC}_\omega(C, \tau, \varkappa)$ is bounded from above by κC. Furthermore, $\mathrm{DC}_\omega(\tau, \varkappa)$ is of full Lebesgue measure.*

Proof. Let $I_{\nu,k}$ be the interval of θ satisfying
$$|\omega \cdot \nu + k\theta| \le \frac{C}{|\nu|^{d+\tau}|k|^{\varkappa}},$$
for some $\nu \in \mathbb{Z}^d \setminus \{0\}$ and $k \in \mathbb{Z} \setminus \{0\}$. Clearly, $I_{\nu,k}$ has width $2C(|\nu|^{d+\tau}|k|^{1+\varkappa})^{-1}$ and is centered at $k^{-1}\omega \cdot \nu$.

Recall that $\|\nu\| = \max_i |\nu_i|$. Clearly, $\|\nu\| \le |\nu|$. Moreover, for each $n \in \mathbb{N}$, one has the following estimate on the cardinality
$$\#\{\nu \in \mathbb{Z}^d \colon \|\nu\| = n\} \le c_1 n^{d-1}$$
for some $c_1 > 0$, depending on d only. So,
$$\sum_{\nu \ne 0} |\nu|^{-(d+\tau)} \le \sum_{\nu \ne 0} \|\nu\|^{-(d+\tau)} \le c_1 \sum_{n \ge 1} n^{-(\tau+1)}$$
which converges if $\tau > 0$. In addition, for $\varkappa > 0$,
$$\sum_{k \ne 0} |k|^{-(1+\varkappa)} < \infty.$$
Therefore, the Lebesgue measure of $\bigcup_{\nu,k} I_{\nu,k}$ is bounded by κC for some constant $\kappa > 0$, depending on d, τ and $\varkappa$.

The measure of the above set approaches zero when $C \to 0$. The second claim is now immediate. □

3. Renormalization

In this section, we construct the renormalization scheme. We first define the non-resonant and resonant modes of a vector field and construct a change of coordinates that, via the pullback, eliminates non-resonant modes of a vector field. We then perform a scaling of the phase space that can produce some non-resonant modes of the transformed vector field. These two transformations, together with a time rescaling, form a one-step renormalization operator. Finally, we construct the stable manifold for a sequence of renormalizations operators.

3.1. **Resonant cones.** As explained in the introduction, we will perform a transformation of our coordinate system such that $\omega \in \mathbb{R}^d$ takes the form $\omega = (1, 0, \ldots, 0)$. In this coordinate system, the lattice $\mathcal{V}_0$ will be a lattice in $\mathbb{R}^d$ which does not coincide with $\mathbb{Z}^d$.

At each renormalization step, we will perform the following phase space scaling $\mathcal{T}(x, y) = (Tx, y)$, where

$$Tx = \eta^{-1} x_\parallel + \beta x_\perp, \tag{3.1}$$

and $x = x_\parallel + x_\perp$ is the decomposition of x into component $x_\parallel$ parallel and $x_\perp$ perpendicular to ω. Notice that we will not scale the coordinate y, and all the functions that we consider will be periodic in y with period 2π; the lattice dual to $2\pi\mathbb{Z}$, the lattice $\mathcal{N} = \mathbb{Z}$, will be fixed throughout the paper.

Under the scaling, the lattice $\mathcal{V}$ is transforms into $T\mathcal{V}$.

Definition 3.1. *Given $\sigma, K > 0$ and a pair of lattices $\mathcal{V}$ and $\mathcal{N}$, in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, the nonresonant index set I^- is defined as the set of pairs $(v, k) \in (\mathcal{V}, \mathcal{N})$ such that $|\omega \cdot v| > \sigma |v|$ and $|k| \le K$, or $v = 0$ and $0 < |k| \le K$. The resonant index set I^+ is defined as the complement of I^- in $\mathcal{V} \times \mathcal{N}$.*

Given any $L \ge 1$, we can find $\ell > 0$ such that

$$|v_\perp| > L \quad \text{or} \quad |v_\parallel| \ge \ell\,, \qquad \forall v \in \mathcal{V} \setminus \{0\}\,. \tag{3.2}$$

We assume that the renormalization parameters $\sigma, \eta, \beta, L, \ell$ are positive and that the following conditions are satisfied

$$\sigma < 1/2\,, \qquad 2\sigma L \le \ell\,, \qquad 0 < \eta \le \beta < 1. \tag{3.3}$$

Given $K > 0$, let

$$J = \{(v, k) \in I^- \,:\, |\theta k| > (1/2)|\omega \cdot v| \text{ and } |k| \le K\}. \tag{3.4}$$

Let

$$\gamma = \max_{(v,k)\in J} \left\{ 2, \theta^{-1}, \frac{\sigma + \sigma|v| + |k|}{|\omega \cdot v + \theta k|} \right\} .$$

Proposition 3.2. *For all modes indexed by (v,k) with $v \neq 0$, $|k| \leq K$ and $|\omega \cdot v| > \sigma|v|$, or $v = 0$ and $0 < |k| \leq K$, if $|\omega \cdot v + \theta k| \neq 0$, then $|\omega \cdot v + \theta k| \geq \sigma/\gamma$, $|\omega \cdot v + \theta k| \geq (\sigma/\gamma)|v|$, and $|\omega \cdot v + \theta k| \geq k/\gamma$.*

Proof. If $|\omega \cdot v| > \sigma|v|$ and $|\theta k| \leq (1/2)|\omega \cdot v|$, then we have $|\omega \cdot v + \theta k| \geq |\omega \cdot v| - |\theta k| \geq (1/2)|\omega \cdot v|$ and thus $|\omega \cdot v + \theta k| \geq (\sigma/2)|v|$. Furthermore, $|\omega \cdot v + \theta k| \geq |\theta k|$. Using the conditions (3.2) and (3.3), together with $L \geq 1$, we also obtain $|\omega \cdot v| > \sigma$ and, thus, $|\omega \cdot v + \theta k| \geq \sigma/2$, in that case.

The number of modes with $|\theta k| > (1/2)|\omega \cdot v| \geq (1/2)\sigma|v|$ and $|k| \leq K$ is finite. So, if $|\omega \cdot v + \theta k| \neq 0$ then $|\omega \cdot v + \theta k| \geq \sigma/\gamma$ and $|\omega \cdot v + \theta k| \geq (\sigma/\gamma)|v|$. □

Let $\mathbb{I}^-$ be the projection operator onto the subspace spanned by modes $(v,k) \in I^-$ defined by the truncation

$$\mathbb{I}^- f(x,y) = \sum_{(v,k)\in I^-} f_{v,k} e^{ix\cdot v + iyk}.$$

The projection operator onto the subspace spanned by modes $(v,k) \in I^+$ is denoted by $\mathbb{I}^+$, and defined as $\mathbb{I}^+ = \mathbb{I} - \mathbb{I}^-$.

3.2. **Elimination of non-resonant modes.** In this subsection, we construct a coordinate transformation $\mathcal{U} = U_1$ such that $\mathcal{U}^* f$ has no non-resonant modes. We construct this transformation using a homotopy method, which is different from the method used in [10, 11, 12, 13, 14, 15]. Let $\omega \in \mathbb{R}^d$, $\theta \in \mathbb{R}$, $\sigma > 0$, $\gamma > 0$, and

$$\epsilon = \frac{\sigma^2}{96\gamma^2(\|(\omega,\theta)\| + 2/3)} \tag{3.5}$$

Theorem 3.3. *Let $\omega \in \mathbb{R}^d$, $\theta \in \mathbb{R}$, $\rho > 0$, $0 < r' < r$, $\sigma > 0$ and $\gamma > 0$. Assume that $0 < \frac{\sigma}{2\gamma} < r - r' < 1$. If $X = (\omega, f)$ with $\|f - \theta\|_{\rho,r} \leq \epsilon$, there is an isotopy $U_t \colon D_{\rho,r'} \to D_{\rho,r}$ of real-analytic diffeomorphisms of the form $U_t(x,y) = (x, y + u_t(x,y))$ such that $U_0 = I$ is the identity map, and*

$$\mathbb{I}^- U_t^* f = (1-t)\mathbb{I}^- f, \qquad t \in [0,1],$$

satisfying

$$\begin{aligned}
\|u_t\|'_{\rho,r'} &\le 4t\gamma\sigma^{-1}\|\mathbb{I}^- f\|_{\rho,r'},\\
\|U_t^* f-\theta\|_{\rho,r'} &\le \left(2+\frac{t}{3}\right)\|f-\theta\|_{\rho,r},\\
\|(\mathbb{I}-\mathbb{E})U_t^* f\|_{\rho,r'} &\le \left(2+\frac{t}{3}\right)\|(\mathbb{I}-\mathbb{E})f\|_{\rho,r}\\
\|\mathbb{E}U_t^* f-\mathbb{E}f\|_{\rho,r'} &\le 8t\chi\gamma^2\sigma^{-2}\|(\mathbb{I}-\mathbb{E})f\|^2_{\rho,r},
\end{aligned} \tag{3.6}$$

where $\chi=\left(4t\|(\omega,\theta)\|+\frac{\sigma}{\gamma}+\frac{1}{r-r'}\frac{\sigma^2}{\gamma^2}\right)$. *Moreover, the map* $f\mapsto U_t$ *is analytic.*

Proof. Define the operator $\mathcal{F}\colon \mathbb{I}^-\mathcal{A}'_{\rho,r'}\to\mathbb{I}^-\mathcal{A}_{\rho,r'}$ as

$$\mathcal{F}(u)=\mathbb{I}^- U^* f=\mathbb{I}^-\frac{-\omega\cdot\partial_x u+f\circ U}{1+\partial_y u}\quad\text{for}\quad \|u\|'_{\rho,r'}<1, \tag{3.7}$$

where $U=I+u$. The derivative of this operator is given by

$$D\mathcal{F}(u)\,h=\mathbb{I}^-\frac{1}{1+\partial_y u}\left(-D_{\omega,\theta}h+\partial_y f\circ U\,h+\frac{D_{\omega,\theta}u-f\circ U+\theta}{1+\partial_y u}\,\partial_y h\right),$$

where $D_{\omega,\theta}=(\omega,\theta)\cdot(\partial_x,\partial_y)$ and the dot denotes the dot product. We would like to determine a one-parameter family u_t, with $0\le t\le 1$, satisfying $\mathcal{F}(u_t)=(1-t)\mathcal{F}(u_0)$ and $u_0=0$.

Firstly, we will show that $D\mathcal{F}(0)=\mathbb{I}^-(-D_{\omega,\theta}+\partial_y f-(f-\theta)\partial_y)$ is invertible. Since

$$\begin{aligned}
\|D^{-1}_{\omega,\theta}\,\mathbb{I}^- h\|'_{\rho,r'} &= \sum_{(v,k)\in I^-}\frac{(1+\|v\|+|k|)\,|h_{v,k}|}{|\omega\cdot v+k\theta|}e^{\rho|v|+r'|k|}\\
&\le \sum_{(v,k)\in I^-}\gamma\sigma^{-1}\,|h_{v,k}|e^{\rho|v|+r'|k|}\\
&\le \gamma\sigma^{-1}\|\mathbb{I}^- h\|_{\rho,r'},
\end{aligned}$$

we obtain that $D^{-1}_{\omega,\theta}\colon \mathbb{I}^-\mathcal{A}_{\rho,r'}\to\mathbb{I}^-\mathcal{A}'_{\rho,r'}$ is well-defined with

$$\|D^{-1}_{\omega,\theta}\|\le\gamma\sigma^{-1}.$$

Since the linear operator $\widehat{f}=\partial_y f-(f-\theta)\partial_y\colon \mathcal{A}'_{\rho,r'}\to\mathcal{A}_{\rho,r'}$ is continuous with norm bounded from above by $\|\widehat{f}\|\le\frac{1}{r-r'}\|f-\theta\|_{\rho,r}+\|f-\theta\|_{\rho,r'}$, the norm of $D\mathcal{F}(0)^{-1}\colon \mathbb{I}^-\mathcal{A}_{\rho,r'}\to\mathbb{I}^-\mathcal{A}'_{\rho,r'}$ is bounded as

$$\|D\mathcal{F}(0)^{-1}\|=\|D^{-1}_{\omega,\theta}(\mathbb{I}-\mathbb{I}^-\widehat{f}D^{-1}_{\omega,\theta})^{-1}\|\le\frac{\|D^{-1}_{\omega,\theta}\|}{1-2\|\widehat{f}\|\|D^{-1}_{\omega,\theta}\|}\le 2\|D^{-1}_{\omega,\theta}\|,$$

for $\|\widehat{f}\| \le (4\|D_{\omega,\theta}^{-1}\|)^{-1}$, that holds if $\frac{1}{r-r'}\|f-\theta\|_{\rho,r}+\|f-\theta\|_{\rho,r'} \le \sigma/(4\gamma)$.

Secondly, we prove that $D\mathcal{F}(u)$ is invertible for sufficiently small u. Notice that, if $\|u\|_{\rho,r'} \le (r-r')/2$, we have

$$\begin{aligned}
\|[D\mathcal{F}(u) - D\mathcal{F}(0)]\, h\|_{\rho,r'} = & \Bigg\| \mathbb{I}^- \frac{1}{1+\partial_y u} \Big[\partial_y u\, D_{\omega,\theta} h + ((\partial_y f)\circ U - \partial_y f)\cdot \\
& \cdot (1+\partial_y u)\, h + (f-\theta)\partial_y u\, \partial_y h - (f\circ U - f)\,\partial_y h \\
& + \frac{D_{\omega,\theta} u + (f-\theta)\circ U \partial_y u}{1+\partial_y u}\, \partial_y h \Big] \Bigg\|_{\rho,r'} \\
\le & \frac{\|u\|'_{\rho,r'}\|h\|'_{\rho,r'}}{1-\|u\|'_{\rho,r'}} \, [\|(\omega,\theta)\| \\
& + \left(1 + \frac{8}{(r-r')^2}(1 + \frac{r-r'}{4} + \|u\|'_{\rho,r'})\right) \\
& \cdot \|f-\theta\|_{\rho,r} + \frac{\|(\omega,\theta)\| + \|f-\theta\|_{\rho,r}}{1-\|u\|'_{\rho,r'}} \Bigg].
\end{aligned}$$

If $r-r'<1$, $\|u\|'_{\rho,r'} \le 1/2$ and $\Delta = 3\|(\omega,\theta)\| + (3+14(r-r')^{-2})\|f-\theta\|'_{\rho,r}$, we obtain

$$\|D\mathcal{F}(u) - D\mathcal{F}(0)\| \le 2\Delta \|u\|'_{\rho,r'}. \tag{3.8}$$

Therefore, if

$$\|u\|'_{\rho,r'} \le \min\left\{\frac{1}{2}, \frac{r-r'}{2}, \frac{1}{4\Delta\|D\mathcal{F}(0)^{-1}\|}\right\} := \delta, \tag{3.9}$$

$D\mathcal{F}(u)$ is invertible, with

$$\|D\mathcal{F}(u)^{-1}\| \le \frac{1}{\|D\mathcal{F}(0)^{-1}\|^{-1} - \|D\mathcal{F}(u) - D\mathcal{F}(0)\|} \le 2\|D\mathcal{F}(0)^{-1}\|.$$

Finally, by first differentiating $\mathcal{F}(u_t) = (1-t)\mathcal{F}(u_0)$ with respect to t and then integrating, we obtain

$$u_t = -\int_0^t D\mathcal{F}(u_s)^{-1}\mathcal{F}(0)\, ds,$$

whenever the family members u_s satisfy the same smallness condition (3.9) as u above so that the derivative of $\mathcal{F}$ is invertible. Furthermore, $u_t \in \mathbb{I}^-\mathcal{A}'_{\rho,r'}$ is real-analytic for each t and satisfies

$$\begin{aligned}
\|u_t\|'_{\rho,r'} & \le t \sup_{\|u\|'_{\rho,r'} \le \delta} \|D\mathcal{F}(u)^{-1}\| \, \|\mathbb{I}^- f\|_{\rho,r'} \\
& \le 2t\|D\mathcal{F}(0)^{-1}\| \, \|\mathbb{I}^- f\|_{\rho,r'} \le t\delta,
\end{aligned}$$

if $\|\mathbb{I}^- f\|_{\rho,r'} \le \delta(2\|D\mathcal{F}(0)^{-1}\|)^{-1}$. The map $f \mapsto u_t$ is analytic.

Since

$$\begin{aligned} U_t^* f - \theta = & - D_{\omega,\theta} u_t + (f-\theta)\circ U_t \\ & + \sum_{n\ge 1} (-\partial_y u_t)^n (-D_{\omega,\theta} u_t + (f-\theta)\circ U_t), \end{aligned} \tag{3.10}$$

and $u_t \in \mathbb{I}^- \mathcal{A}'_{\rho,r'}$, by taking the $\mathbb{I}^+$ projection of (3.10),

$$\mathbb{I}^+ U_t^* f - \theta = \mathbb{I}^+ \left[(f-\theta)\circ U_t + \sum_{n\ge 1} (-\partial_y u_t)^n (-D_{\omega,\theta} u_t + (f-\theta)\circ U_t) \right], \tag{3.11}$$

and assuming $\|f-\theta\|_{\rho,r} \le \sigma/(8\gamma)$, we obtain

$$\begin{aligned} \|\mathbb{I}^+(U_t^* f - \theta)\|_{\rho,r'} & \le \|f-\theta\|_{\rho,r} + \frac{\|u_t\|'_{\rho,r'}}{1-\|u_t\|'_{\rho,r'}} \left(\|u_t\|'_{\rho,r'} \|(\omega,\theta)\| + \|f-\theta\|_{\rho,r} \right) \\ & \le \|f-\theta\|_{\rho,r} + 4t\|\mathbb{I}^- f\|_{\rho,r'} \left(\frac{1}{12} + \|D\mathcal{F}(0)^{-1}\| \|f-\theta\|_{\rho,r} \right) \\ & \le \|f-\theta\|_{\rho,r} + \frac{4}{3} t \|\mathbb{I}^- f\|_{\rho,r'}. \end{aligned}$$

Since, by construction,

$$\mathbb{I}^- U_t^* f = (1-t)\mathbb{I}^- f, \tag{3.12}$$

we also have $\|\mathbb{I}^- U_t^* f\|_{\rho,r'} = (1-t)\|\mathbb{I}^- f\|_{\rho,r'}$, and

$$\|U_t^* f - \theta\|_{\rho,r'} \le \|f-\theta\|_{\rho,r} + \left(1 + \frac{t}{3}\right) \|\mathbb{I}^- f\|_{\rho,r'}.$$

The second inequality in (3.6) follows.

By taking the $\mathbb{I}-\mathbb{E}$ projection of the identities (3.13) and (3.12), adding them up and using the fact that $(\mathbb{I}-\mathbb{E})(f\circ U_t) = (\mathbb{I}-\mathbb{E})((f-\mathbb{E}f)\circ U_t)$, we similarly obtain the third inequality in (3.6).

By taking the $\mathbb{E}$ projection of identity (3.13), we obtain

$$\mathbb{E}U_t^* f - \mathbb{E}f = \mathbb{E}((\mathbb{I}-\mathbb{E})f)\circ U_t + \sum_{n\ge 1} (-\partial_y u_t)^n (-D_{\omega,\theta} u_t + ((\mathbb{I}-\mathbb{E})f)\circ U_t). \tag{3.13}$$

Taking into account that

$$\begin{aligned} \|\mathbb{E}((\mathbb{I}-\mathbb{E})f)\circ U_t\|_{\rho,r'} & \le \|\partial_y((\mathbb{I}-\mathbb{E})f)\|_{\rho,\frac{r+r'}{2}} \|u\|_{\rho,r'} \\ & \le \frac{2}{r-r'} \|(\mathbb{I}-\mathbb{E})f\|_{\rho,r} \|u\|_{\rho,r'}, \end{aligned} \tag{3.14}$$

we obtain the fourth inequality in (3.6). $\square$

3.3. **Phase space and time rescaling.** Consider the linear coordinate transformation $\mathcal{T}\colon (x,y)\mapsto (Tx,y)$ rescaling the base torus, where $T\in \mathrm{SL}(d,\mathbb{R})$ is a matrix associated to ω, defined as in (3.1). In addition, we will perform a linear time rescaling $t\mapsto \eta^{-1}t$.

Since $\omega = \eta^{-1}T^{-1}\omega$, the joint action of $\mathcal{T}$ and time rescaling on $X=(\omega,f)$ is given by

$$\eta^{-1}\mathcal{T}^*X = (\omega, \eta^{-1} f\circ \mathcal{T}).$$

We are interested in the action of this transformation on vector fields with no non-resonant modes, since the non-resonant modes are eliminated by a coordinate change constructed in Theorem 3.3.

Lemma 3.4. *If $0<\rho''\le \eta\rho'$ and $0<\eta,\beta<1$, $0<r''\le r'-\sigma/2$, $\sigma>0$, then $\mathcal{T}^*$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho',r'}(\mathcal{V})$ to $\mathcal{A}_{\rho'',r''}(T\mathcal{V})$, with the property that*

$$\|\mathcal{T}^*\mathbb{I}^+\mathbb{I}_K(\mathbb{I}-\mathbb{E})f\|_{\rho'',r''}\le e^{-\rho'(1-\eta\beta)L}\|\mathbb{I}^+\mathbb{I}_K(\mathbb{I}-\mathbb{E})f\|_{\rho',r'}\,.$$
$$\|\mathcal{T}^*(\mathbb{I}-\mathbb{I}_K)f\|_{\rho'',r''}\le e^{-\frac{1}{2}\sigma K}\|(\mathbb{I}-\mathbb{I}_K)f\|_{\rho',r'}\,.$$
$$\|\mathcal{T}^*\mathbb{E}f\|_{\rho'',r''}\le \|\mathbb{E}f\|_{\rho',r'}\,,$$

Proof. Due to our choice of the norm, it suffices to verify the given bounds for a single mode $\hat f_{v,k}(x,y)=f_{v,k}e^{ix\cdot v+iyk}$ labeled by (v,k). From the definitions of the scaling map it follows that

$$\|\mathcal{T}^*\hat f_{v,k}\|_{\rho'',r''}\le e^{A}\|\hat f_{v,k}\|_{\rho',r'}\,,$$

where $A\le \rho''|Tv_{\|}|+\rho''|Tv_{\perp}|-\rho'|v_{\|}|-\rho'|v_{\perp}|-(r'-r'')|k|$.

In order to prove the first bound, assume that (v,k) belongs to I^+ and $|k|\le K$. Thus, $|v_{\|}|\le\sigma|v|$, with $v\ne 0$. Since $|v|=|v_{\|}|+|v_{\perp}|$, $Tv_{\|}=\eta^{-1}v_{\|}$ and $Tv_{\perp}=\beta v_{\perp}$, we find that $A\le -\rho'(1-\eta\beta)|v_{\perp}|$. Notice now that, in this case, $|v_{\|}|<2\sigma|v_{\perp}|$, by using that $\sigma<1/2$, which does not allow frequencies v that satisfy $|v_{\|}|\le L$ and $|v_{\perp}|\ge \ell$, due to the condition (3.3). Thus, we must have $|v_{\|}|>L$, by condition (3.2).

The second bound follows directly from our initial estimate on A by using that $|Tv|\le|\eta^{-1}v|$ and $|k|>K$. Setting $v=0$ and $k=0$ leads the third bound. □

3.4. **Renormalization transformations.** Following [12, 13, 15], we express the Brjuno condition on ω (and, thus, on $\mathcal{V}$) in terms of the summability of the series of numbers

$$a_n=\sum_{k=n}^{\infty}2^{n-k}\Big[2^{-k-\kappa}\ln(1/\Omega'_{k+\kappa})+(k+\kappa')^{-2}\Big],\qquad \Omega'_n=\min_{0<|\nu_{\perp}|<2^n}|\nu_{\|}|\,, \tag{3.15}$$

for all positive integers n. Here κ and κ' are two integer constants that will be specified later on.

It follows from the definition that $a_{n+1}/2 < a_n < 2a_{n+1}$, for all $n \in \mathbb{N}$ and, thus, $a_{n+1}2^{n+1}/4 < a_n 2^n < a_{n+1}2^{n+1}$. In particular, $a_n 2^n$ is an increasing sequence.

We will define the scaling parameters as in [13],

$$\eta_n = \left(\frac{A_{n+1}}{A_n}\right)^{\frac{d-1}{d}}, \beta_n = \left(\frac{A_{n+1}}{A_n}\right)^{\frac{1}{d}}, \text{ where } A_n = \sum_{k=n}^{\infty} a_k\,, \tag{3.16}$$

for all positive integers n. Since $\{a_n\}_{n\in\mathbb{N}}$ is a summable sequence of positive numbers, the sequence $\{A_n\}_{n\in\mathbb{N}}$ is well-defined, decreasing and converging to zero. We define, recursively, $\lambda_n = \eta_n \lambda_{n-1}$, with $\lambda_0 = 1$. These definitions imply the last bound in (3.3), since $\eta_n < \beta_n < 1$ for $d > 1$.

These parameters are used to define the scaling maps T_n and $P_n = T_n \cdots T_1$, at the n-th renormalization step, for each $n \in \mathbb{N}$, as

$$T_n(x) = \eta_n^{-1} x_\parallel + \beta_n x_\perp\,, \qquad P_n(x) = \lambda_n^{-1} x_\parallel + \big(\prod_{i=1}^{n} \beta_i\big) x_\perp\,. \tag{3.17}$$

We also define $T_0 = P_0$ as the identity maps. Notice that the determinants $|T_n| = |P_n| = 1$, for all $n \in \mathbb{N}$, by the choice of the scaling parameters.

Given a lattice $\mathcal{V}_0 = \mathcal{V} \subset \mathbb{R}^d$, we define the lattice $\mathcal{V}_{n-1} = P_{n-1}\mathcal{V}_0$, defining the frequency space of the functions that are going to be renormalized in the n-th step. The parameters L and ℓ used in the n-th renormalization step are

$$L_{n-1} = 2^{n+\kappa} \prod_{i=1}^{n-1} \beta_i\,, \qquad \ell_{n-1} = \lambda_{n-1}^{-1} e^{-a_n 2^{n+\kappa}}\,. \tag{3.18}$$

Proposition 3.5. *If $v \in \mathcal{V}_{n-1}$ is nonzero, then either $|v_\parallel| \geq \ell_{n-1}$ or $|v_\perp| > L_{n-1}$.*

Proof. Assume that $v \in \mathcal{V}_{n-1}$ satisfies $0 < |v_\perp| \leq L_{n-1}$. Then the corresponding lattice point $\nu = P_{n-1}^{-1} v$ in $\mathcal{V}_0$ satisfies $|\nu_\perp| \leq (\prod_{i=1}^{n-1} \beta_i)^{-1} L_{n-1} = 2^{n+\kappa}$ and, thus, $|\nu_\parallel| \geq \Omega'_{n+\kappa}$ by (3.15). Since we have $\Omega'_{n+\kappa} > e^{-a_n 2^{n+\kappa}}$, this yields

$$|v_\parallel| = \lambda_{n-1}^{-1} |\nu_\parallel| \geq \lambda_{n-1}^{-1} \Omega'_{n+\kappa} > \lambda_{n-1}^{-1} e^{-a_n 2^{n+\kappa}} = \ell_{n-1}\,, \tag{3.19}$$

as claimed. □

Let C_θ be a constant dependent of θ, that will be specified later on.

Definition 3.6. *We define the resonant cone width parameter at the n-th renormalization step*

$$\sigma_n = (2\kappa' C_\theta^{-1}\lambda_{n-1}L_{n-1})^{-1}e^{-a_n 2^{n+\kappa}} = \frac{C_\theta A_1}{2\kappa' A_n}2^{-(n+\kappa)}e^{-a_n 2^{n+\kappa}}. \quad (3.20)$$

This definition immediately implies $\sigma_n > 0$ and $2\sigma_n L_{n-1} \leq \ell_{n-1}$, for $\kappa' > C_\theta$ and all $n \in \mathbb{N}$.

Definition 3.7. *Given the initial domain parameters $\varrho, r > 0$, we define the n-th step cut-off parameter*

$$K_{n-1} = \frac{2\varrho}{A_1\sigma_n}a_n 2^{n+\kappa}. \quad (3.21)$$

The following proposition completes the verification of all bounds in (3.3).

Proposition 3.8. *For any fixed κ' and κ sufficiently large (depending on κ'), we have $\sum_{n=1}^{\infty}\sigma_n < 1/2$.*

Proof. Notice that

$$\sigma_n < \frac{C_\theta A_1}{2\kappa' a_n}2^{-(n+\kappa)}e^{-a_n 2^{n+\kappa}}.$$

Since $\{a_n 2^n\}_{n\in\mathbb{N}}$ is a growing sequence, the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is decreasing. Notice also that for a fixed κ', and sufficiently large κ, we have $2^{n+\kappa}a_n \geq 2^{n+\kappa}(n+\kappa')^{-2} \geq c' 2^\kappa n$, for some constant $c' > 0$ depending only on κ'. This makes the sum $\sum_{n=1}^{\infty}\sigma_n$ finite and, since by choosing κ sufficiently large, A_1 decreases, we can make this sum smaller than $1/2$. □

Definition 3.9. *The initial domain parameters are $\rho_0 = \varrho > 0$ and $r_0 = r > 0$. The n-th step domain parameters are*

$$\rho_{n-1} = \lambda_{n-1}\varrho, \qquad r_{n-1} = r\left[1 - \sum_{i=1}^{n-1}\left(\frac{\sigma_i}{\gamma_i} + \frac{\sigma_i}{2}\right)\right]. \quad (3.22)$$

Remark 3.10. It follows from Proposition 3.8 and the fact that $\gamma_i \geq 2$ that $r_n > r/2$.

Definition 3.11. *Let $\theta_{n-1} = \lambda_{n-1}^{-1}\theta$, for $n \in \mathbb{N}$. Let also*

$$J_{n-1}^{-} = \{(v,k) \in I^{-}(\mathcal{V}_{n-1}) : |\theta_{n-1}k| > (1/2)|\omega\cdot v| \text{ and } k \leq K_{n-1}\},$$

and

$$\gamma_n = \max_{(v,k)\in J_{n-1}^{-}}\left\{2, \theta_{n-1}^{-1}, \frac{\sigma_n + \sigma_n|v| + k}{|\omega\cdot v + \theta_{n-1}k|}\right\}.$$

Definition 3.12. *We say that θ is $DC_\omega(\mathcal{V})$ if there exist constants $\tau, \varkappa > 0$ and $\mathcal{C} > 0$ such that*

$$|\omega \cdot v + \theta k| > \frac{\mathcal{C}}{|v|^{d+\tau}|k|^{\varkappa}}\,, \tag{3.23}$$

for all $v \in \mathcal{V}\backslash\{0\}$ and all $k \in \mathcal{N}$.

In the following, for any $\theta \in DC_\omega(\mathcal{V})$, $\tau, \varkappa, \mathcal{C}$ are the associated constants as in Definition 3.12.

Proposition 3.13. *If θ is a positive number such that $\theta \in DC_\omega(\mathcal{V})$, then there exists a universal constant $\xi > 0$, such that for all $n \in \mathbb{N}$,*

$$\gamma_n < \xi C_\theta(\lambda_{n-1}\sigma_n)^{-(d+\tau)}(\prod_{i=1}^{n-1}\beta_i)^{-(d+\tau)}K_{n-1}^{d+\tau+\varkappa+1}, \tag{3.24}$$

where $C_\theta = \max\{\theta^{-1}, \mathcal{C}^{-1}(\theta+1)\theta^{d+\tau}\}$.

Proof. Recall that $\nu = P_{n-1}^{-1}v$. By definition, we have

$$\begin{aligned}
\gamma_n &= \max_{(v,k)\in J_{n-1}^-}\left\{2, \theta_{n-1}^{-1}, \frac{\sigma_n + \sigma_n|v| + |k|}{|\omega\cdot v + \theta_{n-1}k|}\right\} \\
&< \max_{(v,k)\in J_{n-1}^-}\left\{2, \theta_{n-1}^{-1}, \frac{2|\omega\cdot v| + |k|}{|\omega\cdot v + \theta_{n-1}k|}\right\} \\
&< \max_{(v,k)\in J_{n-1}^-}\left\{2, \theta_{n-1}^{-1}, \frac{(\theta_{n-1}+1)|k|}{|\omega\cdot v + \theta_{n-1}k|}\right\} \\
&< \max_{(v,k)\in J_{n-1}^-}\left\{2, \lambda_{n-1}\theta^{-1}, \frac{(\lambda_{n-1}^{-1}\theta+1)|k|}{\lambda_{n-1}^{-1}|\omega\cdot \nu + \theta\kappa|}\right\}.
\end{aligned} \tag{3.25}$$

Now, using the fact that $\theta \in DC_\omega(\mathcal{V})$, we find

$$\frac{1}{|\omega\cdot\nu + \theta k|} \le \mathcal{C}^{-1}|\nu|^{d+\tau}|k|^{\varkappa} \le \mathcal{C}^{-1}(\prod_{i=1}^{n-1}\beta_i)^{-(d+\tau)}|v|^{d+\tau}|k|^{\varkappa}, \tag{3.26}$$

by using that

$$|v| = |v_\parallel| + |v_\perp| = \lambda_{n-1}^{-1}|\nu_\parallel| + (\prod_{i=1}^{n-1}\beta_i)|\nu_\perp| \ge (\prod_{i=1}^{n-1}\beta_i)|\nu|. \tag{3.27}$$

For $(v,k) \in J_{n-1}^-$ and $v \neq 0$, we have $\sigma_n|v| < |\omega\cdot v| < 2\theta_{n-1}|k| \le 2\theta_{n-1}K_{n-1}$ and, thus, $|v| < 2\theta(\lambda_{n-1}\sigma_n)^{-1}K_{n-1}$.

Therefore, we obtain that

$$\gamma_n < \xi C_\theta(\lambda_{n-1}\sigma_n)^{-(d+\tau)}(\prod_{i=1}^{n-1}\beta_i)^{-(d+\tau)}K_{n-1}^{d+\tau+\varkappa+1}, \tag{3.28}$$

where ξ is a universal constant, and $C_\theta = \max\{\theta^{-1}, \mathcal{C}^{-1}(\theta+1)\theta^{d+\tau}\}$. $\square$

Definition 3.14. *For $n \in \mathbb{N}$, let*

$$\mu_n = \exp\Big\{-\varrho\lambda_{n-1}(1-\beta_n\eta_n)L_{n-1}\Big\} = \exp\Big\{-\frac{\varrho}{A_1}a_n 2^{n+\kappa}\Big\}. \tag{3.29}$$

Proposition 3.15. *$\mu_{n+1} < \mu_n < \mu_{n+1}^{1/4}$, for $n \in \mathbb{N}$. Furthermore, given $C, N > 0$, if κ' and then κ are chosen sufficiently large, then for all $n \geq 1$,*

$$\mu_n \leq Ce^{-N2^{n+\kappa}a_n}, \qquad \mu_n \leq C2^{-Nn}, \qquad \mu_n \leq C\left(\frac{A_n}{A_1}\right)^N. \tag{3.30}$$

Proof. Let $C > 0$ and $N > 0$ be arbitrary. Since $a_{n+1}/2 < a_n < 2a_{n+1}$, for all $n \in \mathbb{N}$, we have $a_{n+1}2^{n+1}/4 < a_n 2^n < a_{n+1}2^{n+1}$, and thus $\mu_{n+1} < \mu_n < \mu_{n+1}^{1/4}$. By choosing κ' and κ sufficiently large, we have $1/A_1 \geq N$. Increasing them further, if needed, we obtain the first bound. Keeping κ' fixed, and increasing κ further, if necessary, we obtain the second two bounds in (3.30) by using that $2^{n+\kappa}a_n \geq 2^{n+\kappa}(n+\kappa')^{-2} \geq c'2^\kappa n$, for some positive constant c' depending only on κ'. The same inequality, together with $A_n/A_1 > a_n/A_1 > C^{1/N}e^{-\varrho 2^{n+\kappa}a_n/(NA_1)}$, where the last inequality is valid for sufficiently large κ, implies the third bound in (3.30). $\square$

Proposition 3.15 directly implies the following claim.

Corollary 3.16. *Given any $C, N > 0$, if κ' and κ are chosen sufficiently large, then for all $n \geq 1$,*

$$\begin{gathered} \mu_n \leq C\sigma_n^N, \qquad \mu_n \leq CK_{n-1}^{-N}, \qquad \mu_n \leq C\eta_n^N, \\ \mu_n \leq C\lambda_n^N \leq C\eta_n^N \leq C\beta_n^N, \qquad \mu_n \leq C\gamma_n^{-N}. \end{gathered} \tag{3.31}$$

Proof. In the first and the last inequality we have also used that $\mu_n \leq C\kappa'^{-N}$ for any given $C, N > 0$, if κ is chosen sufficiently large. $\square$

From Theorem 3.3, it follows that there exists a universal constant $R > 0$ such that the n-th step renormalization operator $\mathfrak{R}_n$ is well-defined from an open ball $B_{n-1} \subset \mathcal{A}_{\rho_{n-1},r_{n-1}}(\mathcal{V}_{n-1})$ around θ_{n-1}, of radius $R\sigma_n^2/(\gamma_n^2\theta_{n-1})$, into $\mathcal{A}_{\rho_n,r_n}(\mathcal{V}_n)$. We may choose the domain of the first step renormalization operator to be any ball $D_0 \subset R\sigma_1^2/(\gamma_1^2\theta)$. Notice that the restriction of $\mathfrak{R}_n$ to $\mathbb{E}\mathcal{A}_{\rho_{n-1},r_{n-1}}(\mathcal{V}_{n-1})$ is a linear operator from $\mathbb{E}\mathcal{A}_{\rho_{n-1},r_{n-1}}(\mathcal{V}_{n-1})$ to $\mathbb{E}\mathcal{A}_{\rho_n,r_n}(\mathcal{V}_n)$ that will be denoted by $\mathcal{L}_n$.

The following claim follows directly from Theorem 3.3 and Lemma 3.4.

Theorem 3.17. *There exists $C, R > 0$ such that the n-th step renormalization operator $\mathfrak{R}_n$ is a bounded analytic map from B_{n-1} into*

$\mathcal{A}_{\rho_n,r_n}(\mathcal{V}_n)$, *that satisfies* $\|\mathcal{L}_n^{-1}\| \leq 1$ *and*

$$\|(\mathbb{I}-\mathbb{E})\mathfrak{R}_n(f_{n-1})\|_{\rho_n,r_n} \leq C\eta_n^{-1}\mu_n\|(\mathbb{I}-\mathbb{E})f_{n-1}\|_{\rho_{n-1},r_{n-1}},$$
$$\|\mathbb{E}\mathfrak{R}_n(f_{n-1}) - \mathfrak{R}_n(\mathbb{E}f_{n-1})\|_{\rho_n,r_n} \leq C\lambda_n^{-1}\gamma_n^2\sigma_n^{-2}\|(\mathbb{I}-\mathbb{E})f_{n-1}\|^2_{\rho_{n-1},r_{n-1}}.$$

In what follows, a domain $\mathcal{D}_{n-1}$ for $\mathfrak{R}_n$ is a subset of B_{n-1} described above, which is open in $\mathcal{A}_{\rho_{n-1},r_{n-1}}(\mathcal{V}_{n-1})$ and contains θ_{n-1}. Given a domain $\mathcal{D}_{n-1}$, for each $\mathfrak{R}_n$, the domain $\widetilde{\mathcal{D}}_{n-1}$ of

$$\widetilde{\mathfrak{R}}_n = \mathfrak{R}_n \circ \cdots \circ \mathfrak{R}_1,$$

for $n \in \mathbb{N}$, is defined recursively as the subset of all functions in the domain of $\widetilde{\mathfrak{R}}_{n-1}$ that are mapped by $\widetilde{\mathfrak{R}}_{n-1}$ into the domain $\mathcal{D}_{n-1}$ of $\mathfrak{R}_n$. By Theorem 3.17, these domains are open, non-empty, and the transformations $\widetilde{\mathfrak{R}}_n$ are analytic.

To prove the following theorem, we apply the stable manifold theorem for sequences of mappings between Banach spaces that was proved in [11] (Section 6 therein).

Theorem 3.18. *If κ' and then κ are chosen sufficiently large, then there exist a sequence of domains $\mathcal{D}_{n-1}$ for the transformations $\mathfrak{R}_n$ such that the set $\mathcal{W} = \cap_{n\in\mathbb{N}\cup\{0\}}\widetilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I}-\mathbb{E})\mathcal{D}_0 \to \mathbb{E}\mathcal{D}_0$ satisfying $W(0) = \theta$ and $DW(0) = 0$. For every $f \in \mathcal{W}$, and every $n \in \mathbb{N}$,*

$$\begin{aligned}\|(\mathbb{I}-\mathbb{E})\widetilde{\mathfrak{R}}_n(f)\|_{\rho_n,r_n} &\leq \chi_n^{1/2}\|(\mathbb{I}-\mathbb{E})f\|_{\varrho,r},\\ \|\mathbb{E}\widetilde{\mathfrak{R}}_n(f) - \theta_n\|_{\rho_n,r_n} &\leq \chi_n\|(\mathbb{I}-\mathbb{E})f\|^2_{\varrho,r},\end{aligned} \tag{3.32}$$

where $\chi_n = \prod_{k=1}^n \mu_k$.

Proof. We start by rescaling the transformations $\mathfrak{R}_n$. For every $n \in \mathbb{N}$, let $d_n = d_{n-1}\sigma_{n+1}^2/(\gamma_{n+1}^2\theta_n)$, with $d_0 > 0$ be half of the constant R from Theorem 3.17. Consider the transformations

$$R_n(g) = d_n^{-1}[\mathfrak{R}_n(\theta_{n-1} + d_{n-1}g) - \theta_{n-1}], \tag{3.33}$$

for $n \in \mathbb{N}$. By Theorem 3.17, R_n is analytic and bounded on a ball $\|g\|_{\rho_{n-1},r_{n-1}} < 2$, and satisfies

$$\begin{aligned}\|(\mathbb{I}-\mathbb{E})R_n(g)\|_{\rho_n,r_n} &\leq \varepsilon_n\|(\mathbb{I}-\mathbb{E})g\|_{\rho_{n-1},r_{n-1}},\\ \|\mathbb{E}R_n(g) - R_n(\mathbb{E}g)\|_{\rho_n,r_n} &\leq \varphi_n\|(\mathbb{I}-\mathbb{E})g\|^2_{\rho_{n-1},r_{n-1}},\end{aligned} \tag{3.34}$$

where $\varepsilon_n = C\eta_n^{-1}\sigma_{n+1}^{-2}\gamma_{n+1}^2\theta_n\mu_n$, $\varphi_n = C\eta_n^{-1}\sigma_n^{-2}\sigma_{n+1}^{-2}\gamma_n^2\gamma_{n+1}^2\theta_n\theta_{n-1}$, and $C > 0$. In addition, we have $\|L_n^{-1}\| < 1/4$. We will restrict R_n to the domain $D_{n-1} \subset \mathcal{A}_{\rho_{n-1},r_{n-1}}(\mathcal{V}_{n-1})$ defined by

$$\|\mathbb{E}g\|_{\rho_{n-1},r_{n-1}} < 1, \quad \|(\mathbb{I}-\mathbb{E})g\|_{\rho_{n-1},r_{n-1}} < \delta_{n-1}, \tag{3.35}$$

where $\delta_{n-1} = (6\varphi_n)^{-1}$. By Corollary 3.16, if κ' and κ are chosen sufficiently large, then $C\eta_n^{-1}\sigma_{n+1}^{-2}\gamma_{n+1}^2\theta_n\mu_n^{1/2} \le 1/6$ and

$$C\eta_{n+1}^{-1}\eta_n^{-1}\lambda_{n+1}^{-1}\sigma_{n+2}^{-2}\sigma_{n+1}^{-2}\sigma_n^2\gamma_{n+2}^2\gamma_{n+1}^2\gamma_n^{-2}\theta\mu_n \le 1,$$

and, therefore, $\varepsilon_n \le \mu_n^{1/2}/6$ and $\varepsilon_n\delta_{n-1} \le \delta_n$, for all $n \in \mathbb{N}$. The assumptions of Theorem 6.1 in [11] are now verified with $\vartheta = 1/4$, and the conclusions of this theorem imply the statements of our claim. □

Let $\mathcal{D}_0$ be a domain whose existence is guaranteed by Theorem 3.18.

Theorem 3.19. *If* $\operatorname{rot} f = \theta$, *and* $f \in \mathcal{D}_0$, *then* $f \in \mathcal{W}$.

Proof. If $\operatorname{rot} f = \theta$, let $f = \theta + d_0 g$ and $g_0 = g$ and $g_n = R_n(g_{n-1})$. Lemma 2.3 guarantees that $|\mathbb{E}f_n - \operatorname{rot} f_n| \le \|(\mathbb{I}-\mathbb{E})f_n\|_{\rho_n,r_n}$ and, thus, analogously to (3.34), we also have

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})R_n(g_{n-1})\|_{\rho_n,r_n} &\le \varepsilon_n\|(\mathbb{I}-\mathbb{E})g_{n-1}\|_{\rho_{n-1},r_{n-1}}, \\ \|\mathbb{E}R_n(g_{n-1})\|_{\rho_n,r_n} &\le \varepsilon_n\|(\mathbb{I}-\mathbb{E})g_{n-1}\|_{\rho_{n-1},r_{n-1}}, \end{aligned} \tag{3.36}$$

as long as $g_{n-1} \in D_{n-1}$. Since, $\varepsilon_n\delta_{n-1} \le \delta_n < 1$, if $g \in D_0$, then $g_n \in D_n$, for all $n \in \mathbb{N}$. Here, we have also used that $\operatorname{rot} f_n = \lambda_n^{-1} \operatorname{rot} f$ (see Lemma 2.3). □

4. Analytic conjugacy

We say that a vector field $X = (\omega, f)$ with $f \in \mathcal{A}_{\rho,r}(\mathcal{V})$ is reducible to $Y = (\omega, \theta)$ if there is a continuous embedding $\Gamma_X : D_0 = \mathbb{T}^d \times \mathbb{T}^1 \to D_{\rho,r}$, such that for all $t \in \mathbb{R}$,

$$\phi_X^t \circ \Gamma_X = \Gamma_X \circ \phi_{\omega,\theta}^t, \tag{4.1}$$

where $\phi_{\omega,\theta}$ is the linear flow of the constant vector field $Y = (\omega, \theta)$. We refer to Γ_X as the conjugacy between the flows of X and Y or the reducibility conjugacy for X.

Consider a one-step renormalization operator $\mathcal{R}$ and a vector field X in the domain of $\mathcal{R}$. If F is any map from $\mathbb{T}^d \times \mathbb{T}^1$ into the domain of $\Lambda_X = \mathcal{U}_X \circ \mathcal{T}$, define the map

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \mathcal{T}^{-1}. \tag{4.2}$$

We will restrict our consideration to maps F of the form $F = I + \psi$, where I is the identity and $\psi(x,y) = (0, \hat{\psi}(x,y))$. Notice, that $\mathcal{M}_X$ preserves the form of these maps, since U_X is also of the same form.

Formally, if $\Gamma_{\mathcal{R}(X)}$ is a conjugacy between the flows of $\mathcal{R}(X)$ and $(\omega, \eta^{-1}\theta)$, then $\Gamma_X = \mathcal{M}_X(\Gamma_{\mathcal{R}(X)})$ is a conjugacy between the flows of X and the vector field (ω, θ). This can be seen easily from the identity

$$\Lambda_X \circ \phi_{\mathcal{R}(X)}^{\eta t} = \phi_X^t \circ \Lambda_X. \tag{4.3}$$

Denote by $\mathcal{A}_0(\mathcal{V})$ the Banach space of continuous functions $F : D_0 \to \mathbb{C}^{d+1}$, with frequency module in $\mathcal{V}$, for which the norm $\|F\|_{0,\mathcal{V}} = \sum_{v\in\mathcal{V},k\in\mathcal{N}} \|F_{v,k}\|$ is finite, where $F_{v,k}$ are the Fourier coefficients of F.

Consider now a fixed but arbitrary vector field $X = X_0$ that belongs to the stable manifold $\mathcal{W}$ of our renormalization transformations, described by Theorem 3.18. Let $X_n = \mathcal{R}_n(X_{n-1})$, for $n \geq 1$. In order to simplify the notation, we will write $\mathcal{U}_k$ in place of the map $\mathcal{U}_{X_k}$ associated to the vector field X_k and $\mathcal{M}_{k+1}$ in place of $\mathcal{M}_{X_k}$. Our goal is to construct an appropriate sequence of functions $\Gamma_k \in \mathcal{A}_0(\mathcal{V}_k)$, satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \mathcal{T}_n^{-1}\,, \qquad \Lambda_n = \mathcal{U}_{n-1} \circ \mathcal{T}_n\,, \tag{4.4}$$

for all $n \in \mathbb{N}$. Then, we will show that Γ_0 is the reducibility conjugacy for X_0.

Let us define $\mathcal{B}_n$, for every $n \geq 0$, to be the vector space $\mathcal{A}_0(\mathcal{V}_n)$, equipped with the norm

$$\|\psi\|'_n = s_n^{-1}\|\psi\|_{0,\mathcal{V}_n} = s_n^{-1} \sum_{v\in\mathcal{V}_n,k\in\mathcal{N}} \|\psi_{v,k}\|\,, \qquad s_n = \frac{r}{2\cdot 4^n}\lambda^n\,. \tag{4.5}$$

Denote by B_n the unit ball in $I + \mathcal{B}_n$, centered at the identity function I, and by $B_n/2$ the ball of radius $1/2$ in the same space.

Proposition 4.1. *If κ' and then κ are chosen sufficiently large, then there exists an open neighborhood B of θ in $\mathcal{A}_{\varrho,r}(\mathcal{V}_0)$ such that, for every $X = (\omega, f)$ with $f \in \mathcal{W} \cap B$, and for every $n \in \mathbb{N}$, the map $\mathcal{M}_n$ is well defined and analytic, as a function from B_n to $\mathcal{B}_{n-1}$. Furthermore, $\mathcal{M}_n$ takes values in $B_{n-1}/2$, and $\|D\mathcal{M}_n(F)\| \leq 1/3$, for all $F \in B_n$.*

Proof. Clearly, $\mathcal{M}_n$ is well-defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = I + g + (\mathcal{U}_{n-1} - I) \circ (I + g)\,, \qquad g = \mathcal{T}_n \circ \psi \circ \mathcal{T}_n^{-1}\,, \tag{4.6}$$

where $\psi = F - I$. In order to estimate the norm of $\mathcal{U}_{n-1} - I$, we can apply Theorem 3.3, with $\rho = \rho_{n-1}$ and $r' = r'_{n-1}$, where $r_n \leq r'_{n-1} < r_{n-1}$. By Theorem 3.18,

$$\begin{aligned} \|\mathcal{U}_{n-1} - I\|_{\rho_{n-1},r'_{n-1}} &\leq 4\frac{\gamma_n}{\sigma_n}\|\mathbb{I}^- f_{n-1}\|_{\rho_{n-1},r'_{n-1}} \\ &\leq 4\frac{\gamma_n}{\sigma_n}\chi_{n-1}^{1/2}\|(\mathbb{I} - \mathbb{E})f\|_{\rho,r} \leq \chi_n^{1/11}\,, \end{aligned} \tag{4.7}$$

for all $n > 1$, and for all $f \in \mathcal{W} \cap B$. Here, we have also used Proposition 3.15, and assumed that κ' and then κ have been chosen sufficiently large, and that the neighborhood B of θ has been chosen sufficiently small (depending on κ' and κ). The first estimate in (4.7) and the final bound also hold for $n = 1$.

The composition with $I+g$ in Equation (4.6) can be controlled by Lemma 2.1, using the fact that $\|g\|_{0,\mathcal{V}_{n-1}} \leq \eta_n^{-1} s_n \|\psi\|'_n \leq r'_{n-1}/2$, since $\|\psi\|'_n \leq 1$, as we assume that $F \in B_n$. Using $s_n/s_{n-1} = \eta_n/4$, we obtain $\|g\|'_{n-1} \leq \eta_n^{-1}\eta_n/4 \leq 1/4$. From (4.7) we obtain $\|\mathcal{U}_{n-1} - I\|'_{n-1} \leq s_{n-1}^{-1}\chi_n^{1/11} \leq 1/4$, if κ' and κ have been chosen sufficiently large. These estimates show that $\mathcal{M}_{n-1}$ maps B_n into $B_{n-1}/2$.

Now, we obtain a bound on the norm of the derivative map

$$D\mathcal{M}_n(F)\bar{\psi} = \bar{g} + D(\mathcal{U}_{n-1} - I) \circ (I+g)\bar{g}, \tag{4.8}$$

where $\bar{g} = \mathcal{T}_n \circ \bar{\psi} \circ \mathcal{T}_n^{-1}$. Since $\|g\|_{0,\mathcal{V}_{n-1}} \leq \rho_{n-1}/2$, using the Cauchy estimate on the derivative, we find

$$\|D(\mathcal{U}_{n-1} - I)\|_{\rho_{n-1},\frac{r_{n-1}}{2}} \leq \frac{2}{r_{n-1}} \|\mathcal{U}_{n-1} - I\|_{\rho_{n-1},r_{n-1}}. \tag{4.9}$$

Since $r_{n-1} > r/2$, we obtain a bound on the norm of this derivative analogous to that of (4.7). This, together with the fact that the inclusion map from B_n into B_{n-1} is bounded in norm by $\eta_n/4$, shows that

$$\|D\mathcal{M}_n(F)\bar{\psi}\|'_{n-1} \leq \frac{s_n}{s_{n-1}}\eta_n^{-1}\left(1 + \|D(\mathcal{U}_{n-1} - I)\|_{\rho_{n-1},\frac{r_{n-1}}{2}}\right)\|\bar{\psi}\|'_n, \tag{4.10}$$

and, consequently, $\|D\mathcal{M}_n(F)\| \leq 1/3$, for all $n \in \mathbb{N}$, and $F \in B_n$. □

Below, we will make use of the following estimate on the difference between the flow for $X = (\omega, f)$ and the flow for the constant vector field $Y = (\omega, \theta)$.

Proposition 4.2. *Let $\tau > 0$ and let $X = (\omega, f)$ be a vector field with $f \in \mathcal{A}_{\varrho,r}(\mathcal{V})$, such that $\tau\|f - \theta\|_{\varrho,r} < r' < r$. Then, for all t in the interval $[-\tau, \tau]$,*

$$\|\phi_X^t - \phi_{\omega,\theta}^t\|_{\varrho,r-r'} \leq \|t(f-\theta)\|_{\varrho,r}\,. \tag{4.11}$$

Let ϕ_n be the flow for the vector field X_n. We start with the identity

$$\phi_{n-1}^t \circ \mathcal{M}_n(F) \circ \phi_{\omega,\theta_{n-1}}^{-t} = \mathcal{M}_n\big(\phi_n^{\eta_n t} \circ F \circ \phi_{\omega,\eta_n^{-1}\theta_{n-1}}^{-\eta_n t}\big)\,, \tag{4.12}$$

which follows from the relation (4.3) between the flow of a vector field and the flow of the renormalized vector field.

Proposition 4.3. *Under the same assumptions as in Proposition 4.1, there exists an open neighborhood B of θ in $\mathcal{A}_{\varrho,r}(\mathcal{V}_0)$, such that for every $X = (\omega, f)$ with $f \in \mathcal{W} \cap B$, and for every $n \geq 1$, the function $\phi_n^s \circ F \circ \phi_{\omega,\theta_n}^{-s}$ belongs to B_n, whenever $F \in B_n/2$ and $|s| \leq \chi_n^{-1/6}$.*

Proof. We will use the following easily verifiable identity

$$\phi_n^s \circ F \circ \phi_{\omega,\theta_n}^{-s} = I + \psi \circ \phi_{\omega,\theta_n}^{-s} + \left[\phi_n^s \circ \phi_{\omega,\theta_n}^{-s} - I\right] \circ \left(I + \psi \circ \phi_{\omega,\theta_n}^{-s}\right). \quad (4.13)$$

Since, by assumption, $\|\psi\|_n' \leq 1/2$, and $\|\psi \circ \phi_{\omega,\theta_n}^{-s}\|_{0,\mathcal{V}_n} = \|\psi\|_{0,\mathcal{V}_n}$, we have $\|\psi \circ \phi_{\omega,\theta_n}^{-s}\|_{0,\mathcal{V}_n} \leq s_n/2$.

By Proposition 4.2 and Theorem 3.18, we have the bound

$$\left\|\phi_n^s \circ \phi_{\omega,\theta_n}^{-s} - I\right\|_{\rho_n, r_n - s_n/2} \leq \|s(f_n - \theta_n)\|_{\rho_n, r_n} \leq 2\chi_n^{1/3} \|(\mathbb{I} - \mathbb{E})f\|_{\varrho,r}, \quad (4.14)$$

provided that the right hand side of this inequality is less than $s_n/2$. This is certainly the case if $\|f - \theta\|_{\varrho,r}$ is chosen sufficiently small. The composition in (4.13) is well-defined since $s_n < r_n$.

The third term on the right hand side of (4.13) can be bounded as

$$\|\left[\phi_n^s \circ \phi_{\omega,\theta_n}^{-s} - I\right] \circ \left(I + \psi \circ \phi_{\omega,\theta_n}^{-s}\right)\|_{n-1}' \leq \frac{\eta_n}{2} \chi_n^{1/3} \|(\mathbb{I} - \mathbb{E})f\|_{\varrho,r}, \quad (4.15)$$

which is smaller than $1/2$, for any $n \geq 1$, if f is sufficiently close to θ. The claim follows. □

We will now construct the conjugacy.

Theorem 4.4. *Under the same assumptions as in Proposition 4.1, there exists an open neighborhood B of θ in $\mathcal{A}_{\varrho,r}(\mathcal{V})$, such that the following holds. Given any $X = (\omega, f)$ with $f \in \mathcal{W} \cap B$, and any sequence of functions $F_k \in B_k$, define*

$$\Gamma_{n,k} = \left(\mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_k\right)(F_k), \qquad 0 \leq n < k. \quad (4.16)$$

Then, the limits $\Gamma_n = \lim_{k\to\infty} \Gamma_{n,k}$ exist in $\mathcal{B}_n$, are independent of the choice of $F_0, F_1, \ldots$, and satisfy the identities (4.4). Furthermore, Γ_0 is the conjugacy between X and (ω, θ), and the map $f \mapsto \Gamma_0$ is analytic and bounded on $\mathcal{W} \cap B$.

Proof. By Proposition 4.1, the map $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least $1/2$. Thus, if $1 \leq n < k < k'$, then the difference $\Gamma_{n,k'} - \Gamma_{n,k}$ is bounded in norm by 2^{n-k+1}. This shows that the sequence $k \mapsto \Gamma_{n,k}$ converges in $\mathcal{B}_n$ to a limit Γ_n, which is independent of the choice of the functions F_k. By choosing $F_k = \Gamma_k$ for all k, we obtain the identities (4.4). The analyticity of $f \mapsto \Gamma_0$ follows, via the chain rule, from the analyticity of the maps used in our construction, and from uniform convergence.

In order to prove that Γ_0 conjugates the flow of X to the linear flow of (ω, θ), we will use the identity (4.12). To be more precise, given a real number t, with $|t| < 1$, define $t_n = \lambda_n t$ for all $n \geq 0$. Proposition 4.3 allows us to iterate the identity (4.12), and get the identity

$$\phi_0^t \circ \Gamma_{0,k} \circ \phi_{\omega,\theta}^{-t} = \left(\mathcal{M}_1 \circ \ldots \circ \mathcal{M}_k\right)\left(\phi_k^{t_k} \circ \phi_{\omega,\theta_k}^{-t_k}\right), \quad (4.17)$$

for all $k > 0$. As proved above, the right (and thus left) hand side of this equation converges in $\mathcal{A}_0(\mathcal{V})$ to Γ_0. In addition, $\Gamma_{0,k} \to \Gamma_0$ in $\mathcal{A}_0(\mathcal{V})$, and the convergence is pointwise as well. Thus, since the flow ϕ_0^t is continuous, we have $\phi_0^t \circ \Gamma_0 \circ \phi_{\omega,\theta}^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$, due to the group property of the flow, and the fact that composition with $\phi_{\omega,\theta}^s$ is an isometry on $\mathcal{A}_0(\mathcal{V})$. $\Box$

In what follows, the reducibility conjugacy Γ_0 for the vector field $X = (\omega, f)$ with $f \in \mathcal{W}$ will be denoted by Γ_X. For convenience, we extend the map $f \mapsto \Gamma_X$ to an open neighborhood of θ, by setting $\Gamma_X = \Gamma_{X'}$, where $X' = (\omega, f')$ with $f' = (\mathbb{I} + W)((\mathbb{I} - \mathbb{E})f)$.

Theorem 4.5. *Let $\rho > \varrho + \delta$, $\rho > r + \delta$ and $\delta > 0$. Under the same assumptions as in Proposition 4.1, there exists an open neighborhood B of θ in $\mathcal{A}_\rho(\mathcal{V}_0)$, such that Γ_X has an analytic continuation to $\|\operatorname{Im} x\| < \delta$ and $|\operatorname{Im} y| < \delta$, for each $X = (\omega, f)$ with $f \in B$. With this continuation, the map $f \mapsto \Gamma_X$ defines a bounded analytic map from B to $\mathcal{A}_\delta(\mathcal{V}_0)$.*

Proof. The proof of this theorem is analogous to the proof of Theorem 4.5 of [11]. Consider the translations $R_{q,p}(x, y) = (x+q, y+p)$, with $q \in \mathbb{R}^d$ and $p \in \mathbb{R}$. As before, for vector fields $X = (\omega, f)$, $R_{q,p}^* X$ denotes the pullback under $R_{q,p}$; the corresponding action on f is denoted by $R_{q,p}^* X$. For functions $F : D_0 \to D_{\rho'}$, with $\rho' > 0$, we define $R_{q,p}^* F = R_{q,p}^{-1} \circ F \circ R_{q,p}$. The renormalization operator $\mathcal{R}$ and the maps $\mathcal{M}_X$, defined in (4.2), satisfy

$$\mathcal{R} \circ R_{q,p}^* = R_{T^{-1}q,p}^* \circ \mathcal{R}, \quad \mathcal{M}_{R_{q,p}^* X} = R_{q,p}^* \circ \mathcal{M}_X \circ (R_{T^{-1}q,p}^*)^{-1}. \quad (4.18)$$

Here, we have used the fact that the translations R_u^* are isometries on the spaces $\mathcal{A}_r(\mathcal{V})$ and that the domain of $\mathcal{R}$ is translation invariant. This also implies that the manifold $\mathcal{W}$ is invariant under translations $R_{q,p}^*$, which is used in the second identity in (4.18).

As was explained above, we can extend the function $f \mapsto \Gamma_X$ to vector fields of the form $X = (\omega, f)$ with f in an open neighborhood of θ, via $\Gamma_X = \Gamma_{X'}$, where $X' = (\omega, f')$ and $f' = (\mathbb{I} + W)((\mathbb{I} - \mathbb{E})f)$. If restricted to a sufficiently small open ball $B \subset \mathcal{A}_{\varrho,r}(\mathcal{V}_0)$, centered at θ, the map $f \mapsto \Gamma_X$ is analytic and bounded on the whole B.

The construction of Γ_0 in the proof of Theorem 4.4, together with identities (4.18) and the invariance property $W = W \circ R_{q,p}^*$, shows that $\Gamma_{R_{q,p}^* X} = R_{q,p}^* \Gamma_X$, for all $X = (\omega, f)$ with $f \in B$. Thus, if $q \in \mathbb{R}^d$ and $p \in \mathbb{R}$, then

$$\Gamma_X(q, p) = \big(R_{q,p} \circ \Gamma_{R_{q,p}^* X}\big)(0, 0), \quad (4.19)$$

for $f \in B$. The idea now is to use the analyticity of map $f \mapsto \Gamma_X$, to extend the right hand side of (4.19) to the complex domain $\|\operatorname{Im} q\| < \delta$

and $|\operatorname{Im} p| < \delta$. Choose an open neighborhood B' of θ in $\mathcal{A}_\rho(\mathcal{V}_0)$ such that $R^*_{q,p}B' \subset B$ for all $(q,p) \in \mathbb{C}^{d+1}$ of norm $\delta' = \rho - \varrho$ or less. Then, the right hand side of (4.19), regarded as a function of (f,q,p), is analytic and bounded on the product of B' with the strips $\|\operatorname{Im} q\| < \delta'$ and $|\operatorname{Im} p| < \delta'$. Denoting this function by G, we have $G(f,\cdot) \in \mathcal{A}_\delta(\mathcal{V}_0)$, for all $f \in B'$. The analyticity of $f \mapsto G(f,\cdot)$ is obtained now by using, for instance, a contour integral formula for $(g(t)-g(0)-tg'(0))/t^2$ with $g(t) = G(f + t\widetilde{f},\cdot)$. $\square$

Theorem 3.18 and Theorem 4.5 imply Theorem 1.1.

References

[1] V. I. Arnol'd, Small denominators I: On the mapping of a circle into itself, *Izv. Akad. Nauk. Math. Serie* **25** (1961), 21–86; *Transl. A.M.S. Serie 2* **46** (1965).

[2] V. I. Arnol'd, Proof of A. N. Kolmogorov's theorem on the preservation of quasiperiodic motions under small perturbations of the Hamiltonian, *Usp.Mat.Nauk. SSSR* **18**(5) (1963), 13–40.

[3] A. D. Brjuno, *Analytic form of differential equations I*, Trudy Moskov. Mat. Obshch. **25** (1971), 119-262; Trans. Moscow Math. Soc. **25** (1973), 131–288.

[4] A. D. Brjuno, *Analytic form of differential equations II*, Trudy Moskov. Mat. Obshch. **26** (1972), 199-239; Trans. Moscow Math. Soc. **26** (1974), 199–239.

[5] L. Corsi and G. Gentile, *Oscillator synchronisation under arbitrary quasi-periodic forcing*, Commun. Math. Phys. **316** 2 (2012), 489–529.

[6] G. Gentile, *Resummation of perturbation series and reducibility for Bryuno skew-product flows*, J. Stat. Phys. **125** 2 (2006), 321–361.

[7] G. Gentile, *Degenerate lower-dimensional tori under the Bryuno condition*, Ergodic Theory and Dynam. Systems **27** 2 (2007) 427–457.

[8] M.R. Herman, *Sur la conjugasion differentiable des difféomorphismes du cercle a de rotations*, Publ. Math. Inst. Hautes Etudes Sci. **49** (1979), 5–234.

[9] R. A. Johnson and J. Moser, The rotation number for almost periodic potentials, Comm. Math. Phys. 84 (1982), 403–438.

[10] K. Khanin, J. Lopes Dias, J. Marklof, *Multidimensional continued fractions, dynamic renormalization and KAM theory*, Commun. Math. Phys. **270** (2007), 197–231.

[11] H. Koch and S. Kocić, *Renormalization of vector fields and Diophantine invariant tori*, Ergodic Theory Dynam. Systems **28** 5 (2008), 1559–1585.

[12] H. Koch and S. Kocić, *A renormalization group aproach to quasiperiodic motion with Brjuno frequencies*, Ergodic Theory Dynam. Systems **30** (2010), 1131–1146.

[13] H. Koch and S. Kocić, *A renormalization aproach to lower-dimensional tori with Brjuno frequency vectors*, J. Differential Equations **249** (2010), 1986–2004.

[14] S. Kocić, *Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori*, Nonlinearity **18** (2005), 1–32.

[15] S. Kocić, *Reducibility of skew-product systems with multidimensional Brjuno base flows*, Discrete Contin. Dyn. Syst. A **29** (2011), 261–283.

[16] A. N. Kolmogorov, *On conservation of conditionally periodic motions for a small change in Hamilton's function*, Dokl. Akad. Nauk SSSR **98** (1954), 527–530.

[17] J.Lopes Dias, *A normal form theorem for Brjuno skew-systems through renormalization*, J. Differential Equations **230** (2006), 1–23.

[18] J. Lopes Dias, *Local conjugacy classes for analytic torus flows*, J. Differential Equations **245** (2008) 468–489.

[19] J. Moser, *Convergent Series Expansions for Quasi-Periodic Motions*, Mathematische Annalen **169** (1967), 136–176.

[20] H. Rüssmann, *On the one-dimensional Schrödinger equation with a quasiperiodic potential*, Nonlinear Dynamics (Internat. Conf., New York, 1979) **6** (2001), 90–107, Ann. New York Acad. Sci. 357, New York Acad. Sci., New York, 1980.

[21] J.-C. Yoccoz, *Petits diviseurs en dimension 1*, Astérisque **231** (1995).

[22] J.-C. Yoccoz, *Analytic linearization of circle diffeomorphisms*, in *Dynamical systems and small divisors*, Ed. Marmi and Yoccoz, Lecture Notes in Mathematics **1784** Springer-Verlag (2002).

Department of Mathematics, University of Mississippi, Hume Hall 305, University, MS 38677, USA

E-mail address: skocic@olemiss.edu

Departamento de Matemática, ISEG, Universidade de Lisboa, Rua do Quelhas 6, 1200-781 Lisboa, Portugal

E-mail address: jldias@iseg.utl.pt