A NORMAL FORM THEOREM FOR BRJUNO SKEW-SYSTEMS THROUGH RENORMALIZATION

JOÃO LOPES DIAS

Abstract. We develop a dynamical renormalization method for the problem of local reducibility of analytic linear quasi-periodic skew-product flows on $\mathbb{T}^2 \times \mathrm{SL}(2,\mathbb{R})$. This approach is based on the continued fraction expansion of the linear base flow and deals with 'small-divisors' by turning them into 'large-divisors'. We use this method to give a new proof of a normal form theorem for a Brjuno base flow.

1. Introduction

1.1. **Preliminaries.** Consider the manifold $\mathbb{T}^d \times G$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and G is a Lie group with Lie algebra $\mathfrak{g}$. The transformation of an arbitrary vector field X on $\mathbb{T}^d \times G$ by a diffeomorphism $\psi \colon \mathbb{T}^d \times G \to \mathbb{T}^d \times G$ is given by

$$\psi^* X = D\psi \circ \psi^{-1} \cdot X \circ \psi^{-1}. \tag{1.1}$$

Let V^ω be the set of real-analytic vector fields on this manifold of the form:

$$X(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{\omega}, f(\boldsymbol{x})\,\boldsymbol{y}), \quad (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{T}^d \times G, \tag{1.2}$$

where $\boldsymbol{\omega} \in \mathbb{R}^d - \{0\}$ and $f \in C^\omega(\mathbb{T}^d,\mathfrak{g})$. Each element of V^ω generates a skew-product flow on $\mathbb{T}^d \times G$. That is, at the "base" $\mathbb{T}^d$ we have the linear flow $\boldsymbol{x} \mapsto \boldsymbol{x} + t\boldsymbol{\omega}$, $t \geq 0$, and on the "fiber" G the flow obtained by solving $\dot{\boldsymbol{y}} = f(\boldsymbol{x} + t\boldsymbol{\omega})\,\boldsymbol{y}$.

As we want to preserve the space V^ω under real-analytic coordinate changes, we restrict them to the set D^ω, i.e. of the type

$$\psi(\boldsymbol{x},\boldsymbol{y}) = (T\boldsymbol{x}, F(\boldsymbol{x})\,\boldsymbol{y}), \quad (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{T}^d \times G, \tag{1.3}$$

where $F \in C^\omega(\mathbb{T}^d, G)$ and $T \in \mathrm{SL}(d,\mathbb{Z})$ is a linear automorphism of the torus. A vector field $X \in V^\omega$ in the new coordinates is then given by the formula

$$\psi^* X(\boldsymbol{x},\boldsymbol{y}) = (T\omega, L_{\boldsymbol{\omega}} F(T^{-1}\boldsymbol{x}) \cdot F(T^{-1}\boldsymbol{x})^{-1}\,\boldsymbol{y} + \mathrm{Ad}_{F(T^{-1}\boldsymbol{x})} f(T^{-1}\boldsymbol{x}) \cdot \boldsymbol{y}), \tag{1.4}$$

where

$$L_{\boldsymbol{\omega}} = \boldsymbol{\omega} \cdot D = \sum_i \omega_i \partial/\partial x_i \tag{1.5}$$

and $\mathrm{Ad}_A\, b = AbA^{-1}$ with $A \in G$ and $b \in \mathfrak{g}$.

In some cases it is possible to find a diffeomorphism that simplifies X, in particular reducing it to a "constant" vector field. More precisely, we have the following definition.

Definition 1.1. $X \in V^\omega$ is *C^ω-conjugated to* $Y \in V^\omega$ if there is $\psi \in D^\omega$ such that $\psi^* X = Y$. In case both vector fields have the same rationally independent base vector $\boldsymbol{\omega} \in \mathbb{R}^d - \{0\}$ (i.e. $\boldsymbol{\omega} \cdot \boldsymbol{k} \neq 0$ for all $\boldsymbol{k} \in \mathbb{Z}^d - \{0\}$) and

$$Y(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{\omega}, u\,\boldsymbol{y}), \quad (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{T}^d \times G, \tag{1.6}$$

with $u \in \mathfrak{g}$, then X is said to be *C^ω-reducible to* Y.

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Remark 1.1. Assuming the notation above, a more general definition is the following: X is reducible to Y if it is conjugated to a Y constant along the orbits $\{\boldsymbol{x}+\boldsymbol{\omega}t\}_{t\geq 0}$ for each $\boldsymbol{x}\in\mathbb{T}^d$.

Remark 1.2. The definition of reducibility requires in several contexts (cf. [7]) the use of diffeomorphisms defined on a finite covering $(2\mathbb{T})^d\times G$. In the present work we are able to restrict to the domain $\mathbb{T}^d\times G$.

In this paper we study the $d=2$ case and we choose $G=\mathrm{SL}(2,\mathbb{R})$ and $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$, corresponding to the non-compact Lie group of unimodular real 2×2 matrices and its respective Lie algebra of traceless matrices.

Definition 1.2. A matrix $u\in\mathfrak{sl}(2,\mathbb{R})$ with eigenvalues $\pm 2\pi\mathrm{i}\rho$, $\rho>0$, is *diophantine with respect to* $\boldsymbol{\omega}$ if there exists $C,\tau>0$ such that

$$|2\rho-\boldsymbol{k}\cdot\boldsymbol{\omega}|>\frac{C}{\|\boldsymbol{k}\|^\tau},\quad \mathbb{Z}^d-\{0\}. \tag{1.7}$$

1.2. **Main result.** Even if the goal of this paper is to develop a new method, we use it to restate a 'classical' normal form theorem in the spirit of the work by Dinaburg and Sinai [5]. We prove it (and indeed improve it for the present case) using a novel and considerably simpler approach coming from the renormalization of vector fields appearing in [16, 19, 20, 21].

Theorem 1.3. *Let* $\boldsymbol{\omega}\in\mathbb{R}^2$ *with slope a Brjuno number and* Y *given by* (1.6) *with* u *diophantine with respect to* $\boldsymbol{\omega}$*. If* $X\in V^\omega$ *with base vector* $\boldsymbol{\omega}$ *is sufficiently close to* Y*, there is* $a\in\mathfrak{sl}(2,\mathbb{R})$ *such that the vector field given by*

$$X(\boldsymbol{x},\boldsymbol{y})+(0,a\,\boldsymbol{y}),\qquad(\boldsymbol{x},\boldsymbol{y})\in\mathbb{T}^2\times\mathrm{SL}(2,\mathbb{R}),$$

is C^ω*-reducible to* Y *through a conjugacy* $\psi\in D^\omega$*. Moreover, the map* $X\mapsto(a,\psi)$ *is analytic.*

Remark 1.4. The above theorem can be used to improve the arithmetical condition on the base frequency of [5]'s main result, after performing the steps in chapter 4 of [17]. That is, by looking at the Whitney dependence on u of the counter-term a and the conjugacy, it can be shown the existence of a positive measure set of parameters (Energy) for which the one-dimensional quasi-periodic linear Schrödinger equation is reducible.

The results on reducibility can be seen as extensions of Floquet theory to quasi-periodic linear systems of differential equations. The well-known Floquet theorem states that linear differential equations with time periodic coefficients can always be reduced to constant coefficients systems using a time periodic coordinate transformation. The situation for quasi-periodic systems is more complicated as resonant phenomena appears due to the existence of small-divisors (see e.g. [26]). So, further conditions on the systems have to be imposed in order to achieve the same kind of result. Most of the work on the reducibility for the quasi-periodic systems has been done for the particular case of the quasi-periodic linear 1-dim Schrödinger equation (as is the case of [5]) in both the continuous and discrete-time frameworks. We remark that for u hyperbolic, the reducibility of such systems is known since the 70's [2]. Other cases have also been studied, such as the fiber G being a compact group [17, 18, 29].

Traditional KAM methods use a modified Newton method to construct a sequence of coordinate changes defined on shrinking domains that, in the limit, transforms the problem into the unperturbed case ([5, 6, 25]). In the present paper we shall develop a new method based on the use of the continued fraction expansion of the slope of the frequency vector $\boldsymbol{\omega}$. Its arithmetic properties are reflected on the way we deal with the resonant Fourier modes (corresponding to small divisors), as these can be made non-resonant (large divisors) by the action of the Gauss map. Non-resonant modes are less relevant to the differentiability of the conjugacy, and in fact can be fully eliminated by a non-linear isotopic to the identity coordinate transformation. Those are the main steps for the construction of a renormalization operator acting on the space of vector fields of the form (1.2).

Theorem 1.3 is proved by showing that there is a parameter a for each vector field X close to Y such that the orbit of $X+(0,a\cdot)$ under renormalization is attracted to the orbit of Y. In contrast with previous results of KAM-type in the context of quasi-periodic skew-products, we have managed to improve the condition on the allowed frequencies using the renormalization method, namely generalizing the diophantines to the Brjuno numbers (see section 2.12). A large amount of available KAM-derived results can be improved in this way (e.g. [27]). It should be mentioned here the works of Rüssmann (cf. e.g. [28]) among others, where the Brjuno condition is obtained using KAM methods for systems other than skew-products.

Other renormalization ideas have been used in the context of the stability of invariant tori for nearly integrable Hamiltonian systems [3, 9, 11] and the Melnikov problem for PDE's [4]. They can be viewed as an iterative resummation of the Lindstedt series inspired by quantum field theory and an analogy with KAM theory [8, 10] (recent versions have found the Brjuno conditon for the standard map [1]). Their approach is different from our present method which is essentially dynamical in the sense that it is a dynamical system on the space of vector fields. The action on this space given by the Gauss map, is responsible for the change of scale. The condition on the frequency is hence due to the arithmetic properties of the frequency itself, and not by dealing directly with small divisors. The dynamical renormalization scheme has proved to be a natural and powerful tool in the study of quasi-periodic and small divisor problems of various kind as for instance [14, 15, 22, 24, 29, 30].

1.3. **Outline of the paper.** In section 2 we present the construction of the renormalization method using the continued fractions algorithm. It includes the convergence of the renormalization scheme to a trivial limit set, which in turn implies the existence of the analytic (fibered) conjugacy to a constant system (section 3), i.e. reducibility. In the last part, section 4, we proof one step of the renormalization procedure, namely the elimination of non-resonant modes (avoiding problems related to small-divisors), by means of a homotopy method. Finally, in appendix A we briefly describe a way to adapt the renormalization operator in order to have trivial fixed points, which we show to be hyperbolic.

2. Renormalization of skew-product flows

2.1. **Definitions.** As the tangent bundle of the 2-torus is trivial, $T\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$, we identify the set of vector fields on $\mathbb{T}^2$ with the set of functions from $\mathbb{T}^2$ to $\mathbb{R}^2$, that can be regarded as maps of $\mathbb{R}^2$ by lifting to the universal cover. We will make use of the analyticity to extend to the complex domain, so we will deal with complex analytic functions.

We will be using maps between Banach spaces over $\mathbb{C}$ with a notion of analyticity stated as follows (cf. e.g. [13]): a map F defined on a domain is analytic if it is locally bounded and Gâteaux differentiable. If it is analytic on a domain, it is continuous and Fréchet differentiable. Moreover, we have a convergence theorem which is going to be used later on. Let $\{F_k\}$ be a sequence of functions analytic and uniformly locally bounded on a domain D. If $\lim_{k\to+\infty} F_k = F$ on D, then F is analytic on D.

In the following $A \ll B$ stands for the existence of a constant $C>0$ such that $A \leq CB$.

2.2. **Spaces of analytic skew-products.** Let $r>0$ and consider the domain

$$\mathcal{D}_r = \{\boldsymbol{x} \in \mathbb{C}^2 \colon \|\operatorname{Im} \boldsymbol{x}\| < r/2\pi\} \tag{2.1}$$

for the norm $\|\boldsymbol{z}\| = \sum_i |z_i|$ on $\mathbb{C}^2$. Take a real-analytic map

$$F\colon \mathcal{D}_r \to \mathrm{SL}(2,\mathbb{C}),$$

$\mathbb{Z}^2$-periodic, on the form of the Fourier series

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} F_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}} \tag{2.2}$$

with $F_{\boldsymbol{k}} \in \mathrm{SL}(2,\mathbb{C})$. The Banach spaces $\mathcal{A}_r$ and $\mathcal{A}'_r$ are the subspaces such that the respective norms

$$\|F\|_r = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \|F_{\boldsymbol{k}}\| \, \mathrm{e}^{r\|\boldsymbol{k}\|}, \tag{2.3}$$

$$\|F\|'_r = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+2\pi\|\boldsymbol{k}\|) \, \|F_{\boldsymbol{k}}\| \, \mathrm{e}^{r\|\boldsymbol{k}\|} \tag{2.4}$$

are finite. Here and in the following we use the matrix norm $\|A\| = \max_j \sum_i |A_{i,j}|$ for any square matrix A with entries $A_{i,j}$.

Similarly, define the space $\mathfrak{a}_r$ of real-analytic functions $\mathcal{D}_r \to \mathfrak{sl}(2,\mathbb{C})$, $\mathbb{Z}^2$-periodic and on the form of Fourier series, having the same type of bounded norm as (2.3). We are interested in vector fields that can be written as

$$X(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{\omega}, f(\boldsymbol{x})\,\boldsymbol{y}), \quad (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{D}_r \times \mathrm{SL}(2,\mathbb{C}). \tag{2.5}$$

The space of such vector fields is denoted by V_r whenever f is in $\mathfrak{a}_r$. The norm on this space is defined to be

$$\|X\|_r = \|\boldsymbol{\omega}\| + \|f\|_r. \tag{2.6}$$

2.3. **Continued fractions.** Consider an irrational number $0<\alpha=\alpha_0<1$ written in its continued fractions expansion:

$$\alpha = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}, \tag{2.7}$$

$a_n \in \mathbb{N}$. Its iterates under the Gauss map are $\alpha_n = \{\alpha_{n-1}^{-1}\} = [a_{n+1}, \ldots]$, $n\in\mathbb{N}$, that is

$$\alpha_n = \frac{1}{a_{n+1} + \alpha_{n+1}}. \tag{2.8}$$

Let $\beta_n = \prod_{i=0}^n \alpha_n$, $n \in \mathbb{N}\cup\{0\}$. It is a well-known fact (cf. [12]) that

$$\alpha_n \alpha_{n+1} \leq \gamma^2 \quad \text{and} \quad \beta_n \leq \gamma^n, \tag{2.9}$$

where $\gamma = (\sqrt{5}-1)/2$ is the golden mean.

Consider the transfer matrices in $\mathrm{GL}(2,\mathbb{Z})$:

$$T^{(n)} = \begin{bmatrix} -a_n & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.10}$$

In addition, define $P^{(0)} = I$ and

$$P^{(n)} = T^{(n)} \dots T^{(1)} = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}^{-1}, \quad n \in \mathbb{N}. \tag{2.11}$$

As in [12], this gives the rational approximants $p_n/q_n = [a_1, \dots, a_n] \in \mathbb{Q}$ with

$$p_n - \alpha q_n = (-1)^n \beta_n \quad \text{and} \quad \frac{1}{2q_{n+1}} \le \beta_n \le \frac{1}{q_{n+1}}. \tag{2.12}$$

Hence,

$$\frac{1}{2\beta_{n-1}} \le \|P^{(n)}\| \le \frac{3}{\beta_{n-1}}. \tag{2.13}$$

Finally, define the sequences of vectors in $\mathbb{R}^2$:

$$\begin{aligned} \boldsymbol{\omega}^{(n)} &= \alpha_{n-1}^{-1} T^{(n)} \boldsymbol{\omega}^{(n-1)} = (\alpha_n, 1) \\ \boldsymbol{\Omega}^{(n)} &= -\alpha_{n-1}^{-1}\, {}^{\top}T^{(n)^{-1}} \boldsymbol{\Omega}^{(n-1)} = (1, -\alpha_n). \end{aligned} \tag{2.14}$$

2.4. **Constant modes.** Denote the constant Fourier modes of some f in $\mathfrak{a}_r$ as $\mathbb{E}f \in \mathfrak{sl}(2,\mathbb{R})$. We will use the same notation for the corresponding projection in V_r, i.e.

$$\mathbb{E}X(\boldsymbol{y}) = (\boldsymbol{\omega}, \mathbb{E}f\, \boldsymbol{y}). \tag{2.15}$$

In the following we will be studying vector fields depending on a parameter λ in $\mathfrak{sl}(2,\mathbb{C})$. Take the open ball

$$\mathcal{P}_\mu = \{\lambda \in \mathfrak{sl}(2,\mathbb{C}) \colon \|\lambda\| < \mu\} \tag{2.16}$$

for a given radius $\mu > 0$.

Given $u \in \mathfrak{sl}(2,\mathbb{R})$ with imaginary eigenvalues, there is a unique $\rho > 0$ such that

$$u = 2\pi\rho\, M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M^{-1}$$

for some real matrix M with positive determinant. Define the sequences of matrices in $\mathfrak{sl}(2,\mathbb{R})$ and corresponding eigenvalues:

$$u^{(n)} = \alpha_{n-1}^{-1} u^{(n-1)}, \qquad \rho_n = \alpha_{n-1}^{-1} \rho_{n-1}, \tag{2.17}$$

with $u^{(0)} = u$ and $\rho_0 = \rho$. We will be using the following vector field depending on the parameter λ:

$$Y_\lambda^{(n)}(\boldsymbol{y}) = (\boldsymbol{\omega}^{(n)}, (u^{(n)} + \lambda)\, \boldsymbol{y}). \tag{2.18}$$

2.5. **Change of basis and time rescaling.** The fundamental step of the renormalization is a linear coordinate change of the domain of definition of our vector fields. This is done by a linear transformation derived from the continued fraction expansion of $\boldsymbol{\omega}$. In addition, we perform a linear change of time.

Let $r_{n-1}, \mu_{n-1} > 0$ and for each $\lambda \in \mathcal{P}_{\mu_{n-1}}$ consider the vector field X_λ in $V_{r_{n-1}}$ given by

$$X_\lambda(\boldsymbol{x}, \boldsymbol{y}) = Y_\lambda^{(n-1)}(\boldsymbol{y}) + (0, f_\lambda^{(n-1)}(\boldsymbol{x})\, \boldsymbol{y}), \quad n \in \mathbb{N}. \tag{2.19}$$

Also, $f_\lambda^{(n-1)} \in \mathfrak{a}_{r_{n-1}}$ for each λ, and the dependence on the parameter is real-analytic.

We are interested in the following linear coordinate and time changes:

$$L_n(\boldsymbol{x}, \boldsymbol{y}) = (T^{(n)} \boldsymbol{x}, \boldsymbol{y}), \qquad t \mapsto \alpha_{n-1}^{-1} t. \tag{2.20}$$

The corresponding transformation of the vector field for the new coordinates is $\alpha_{n-1}^{-1} L_n^*$ so that

$$\alpha_{n-1}^{-1} L_n^*(X_\lambda)(\boldsymbol{x}, \boldsymbol{y}) = Y^{(n)}_{\alpha_{n-1}^{-1}\lambda} + (0, \alpha_{n-1}^{-1} f_\lambda^{(n-1)}(T^{(n)^{-1}} \boldsymbol{x})\, \boldsymbol{y}). \tag{2.21}$$

2.6. **Reparametrisation.** We use a change of the parameter λ in order to cancel the perturbation on the constant mode.

Consider the parameter transformation

$$\Lambda_n(\lambda) = \alpha_{n-1}^{-1}(\lambda + \mathbb{E} f_\lambda^{(n-1)}). \tag{2.22}$$

Write

$$S_{n-1}(f_\lambda^{(n-1)}) = \sup_{\lambda \in \mathcal{P}_{\mu_{n-1}}} \|\mathbb{E} f_\lambda^{(n-1)}\| \tag{2.23}$$

and denote by Δ_{n-1} the subset in $\mathfrak{a}_{r-1}$ whose elements f satisfy $S_{n-1}(f) < \frac{\mu_{n-1}}{4}$.

Proposition 2.1. *If*

$$0 < \mu_n \leq \frac{\mu_{n-1}}{4\alpha_{n-1}}, \tag{2.24}$$

then there exists an analytic map $f \mapsto \Lambda_n^{-1}$ from Δ_{n-1} into $\operatorname{Diff}(\mathcal{P}_{\mu_n}, \mathcal{P}_{\mu_{n-1}})$. *If X is real-analytic, then Λ_n^{-1} is also real-analytic.*

Proof. We have (by the Cauchy estimate)

$$\sup_{\mathcal{P}_{\mu_{n-1}/2}} \|D\mathbb{E} f_\lambda\| \leq \frac{2S_{n-1}}{\mu_{n-1}} \leq \frac{1}{2}. \tag{2.25}$$

So, $\lambda \mapsto F(\lambda) = \lambda + \mathbb{E} f_\lambda$ is a diffeomorphism on $\mathcal{P}_{\mu_{n-1}/2}$.

Now, if $\mu < \mu_{n-1}/2 - S_{n-1}$, we have $F^{-1}(\mathcal{P}_\mu) \subset \mathcal{P}_{\mu_{n-1}/2}$. Fix an arbitrary $\zeta \in F^{-1}(\mathcal{P}_\mu)$ and $z = F(\zeta) \in \mathcal{P}_\mu$. By the mean value theorem there is $\xi \in \mathcal{P}_{r_{n-1}/2}$ such that $F^{-1}(z) = DF(\xi)^{-1}(z - F(0))$.

Therefore, $\Lambda_n^{-1} = F^{-1}(\alpha_{n-1}\cdot)$ is a diffeomorphism on $\mathcal{P}_{\mu_n}$ by choosing $\mu \geq \mu_n \alpha_{n-1}$. In addition, $f \mapsto \Lambda_n^{-1}$ is analytic from its dependence on $\mathbb{E} f$. When restricted to a real domain for a real-analytic $\mathbb{E} f$, Λ_n^{-1} is also real-analytic.

Finally, writing $z = \alpha_{n-1} y$ where $y \in \mathcal{P}_{\mu_n}$, we get

$$\begin{aligned} \|\Lambda_n(X)^{-1}(y)\| &\leq \frac{1}{1 - \sup_{\mathcal{P}_{r_{n-1}/2}} \|D\mathbb{E} f\|} \|z - \mathbb{E} f_0\| \\ &\leq 2(\mu_n \alpha_{n-1} + S_{n-1}). \end{aligned} \tag{2.26}$$

□

Define the operator $\mathcal{L}_n$ including the transformation in section 2.5 and the above reparametrisation, given by

$$\begin{aligned} \mathcal{L}_n(X_\lambda)(\boldsymbol{x}, \boldsymbol{y}) &= \alpha_{n-1}^{-1} L_n^*(X_\lambda)(\boldsymbol{x}, \boldsymbol{y}) \\ &= Y_{\Lambda_n(\lambda)}^{(n)}(\boldsymbol{y}) + (0, \alpha_{n-1}^{-1}(\mathbb{I} - \mathbb{E}) f_\lambda^{(n-1)}({T^{(n)}}^{-1}\boldsymbol{x})\, \boldsymbol{y}), \end{aligned} \tag{2.27}$$

for every $(\boldsymbol{x}, \boldsymbol{y}) \in T^{(n)}\mathcal{D}_{r_{n-1}} \times \mathrm{SL}(2, \mathbb{C})$ and $\lambda \in \mathcal{P}_{\mu_{n-1}}$.

2.7. **Resonant cone.**

2.7.1. *Definition.* Given $\sigma_n > 0$, $n \in \mathbb{N} \cup \{0\}$, we define the *resonant modes* with respect to $\boldsymbol{\omega}^{(n)}$ to be the ones whose indices are in the cone

$$I_n^+ = \left\{ \boldsymbol{k} \in \mathbb{Z}^2 \colon |\boldsymbol{\omega}^{(n)} \cdot \boldsymbol{k}| \leq \sigma_n \|\boldsymbol{k}\| \right\}. \tag{2.28}$$

Similarly, the *non-resonant modes* correspond to $I_n^- = \mathbb{Z}^2 - I_n^+$. It is also useful to define the projections $\mathbb{I}_n^+$ and $\mathbb{I}_n^-$ on the above spaces by restricting the Fourier modes to I_n^+ and I_n^-, respectively. The identity operator is $\mathbb{I} = \mathbb{I}_n^+ + \mathbb{I}_n^-$.

2.7.2. *Hyperbolicity of the transfer matrices.* Let

$$A_n = \sigma_n \|T^{(n+1)^{-1}}\| + \alpha_n \frac{\|\mathbf{\Omega}^{(n+1)}\|}{\|\mathbf{\Omega}^{(n)}\|}. \tag{2.29}$$

Lemma 2.1. *For all $\boldsymbol{k} \in I^+_{n-1}$ and $n \in \mathbb{N}$, we have*

$$\| {}^\top T^{(n)^{-1}} \boldsymbol{k}\| \leq A_{n-1} \|\boldsymbol{k}\|. \tag{2.30}$$

Proof. We write $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, where

$$\boldsymbol{k}_1 = \frac{\boldsymbol{k} \cdot \boldsymbol{\omega}^{(n-1)}}{\boldsymbol{\omega}^{(n-1)} \cdot \boldsymbol{\omega}^{(n-1)}} \, \boldsymbol{\omega}^{(n-1)}, \qquad \boldsymbol{k}_2 \in \mathbb{R}\mathbf{\Omega}^{(n-1)}. \tag{2.31}$$

Firstly,

$$\| {}^\top T^{(n)^{-1}} \boldsymbol{k}_1 \| = \frac{\| {}^\top T^{(n)^{-1}} \boldsymbol{\omega}^{(n-1)} \|}{|\boldsymbol{\omega}^{(n-1)} \cdot \boldsymbol{\omega}^{(n-1)}|} |\boldsymbol{k} \cdot \boldsymbol{\omega}^{(n-1)}| \leq \sigma_{n-1} \|T^{(n)^{-1}}\| \, \|\boldsymbol{k}\| \tag{2.32}$$

since $\boldsymbol{k} \in I^+_{n-1}$ and

$$\frac{\| {}^\top T^{(n)^{-1}} \boldsymbol{\omega}^{(n-1)} \|}{|\boldsymbol{\omega}^{(n-1)} \cdot \boldsymbol{\omega}^{(n-1)}|} = \frac{1 + \alpha_{n-1} + a_n}{(1 + \alpha_{n-1}^2) \|T^{(n)^{-1}}\|} \|T^{(n)^{-1}}\| \leq \|T^{(n)^{-1}}\|. \tag{2.33}$$

Secondly, using

$$\| {}^\top T^{(n)^{-1}} \boldsymbol{k}_2 \| = \alpha_{n-1} \frac{\|\mathbf{\Omega}^{(n)}\|}{\|\mathbf{\Omega}^{(n-1)}\|} \|\boldsymbol{k}_2\|, \tag{2.34}$$

we get (2.30). □

2.8. **Estimate on the non-constant resonant modes.** Consider the parameter λ to be fixed. For this reason, in this section we drop it from our notations.

Proposition 2.2. *If*

$$0 < r'_n \leq \frac{r_{n-1}}{A_{n-1}}, \tag{2.35}$$

then $\mathcal{L}_n$ as a map from $(\mathbb{I}^+_{n-1} - \mathbb{E}) V_{r_{n-1}}$ into $(\mathbb{I} - \mathbb{E}) V_{r'_n}$ is continuous with norm estimated from above by α_{n-1}^{-1}.

Proof. Let $f \in (\mathbb{I}^+_{n-1} - \mathbb{E}) \mathfrak{a}_{r_{n-1}}$. Then,

$$\|f \circ T^{(n)^{-1}}\|_{r'_n} = \sum_{\boldsymbol{k} \in I^+_{n-1} - \{0\}} \|f_{\boldsymbol{k}}\| \mathrm{e}^{r'_n \| {}^\top T^{(n)^{-1}} \boldsymbol{k}\|} \leq \|f\|_{r_{n-1}},$$

where we have used (2.30). Finally, writing $X = (0, f \cdot)$, we get

$$\|\mathcal{L}_n(X)\|_{r'_n} \leq \alpha_{n-1}^{-1} \|f \circ T^{(n)^{-1}}\|_{r'_n}.$$

□

2.9. **Analyticity strip cut-off.** Take the inclusion $\mathcal{I}_n \colon V_{r'_n} \to V_{r_n}$ by restricting $X \in V_{r'_n}$ to the smaller domain $\mathcal{D}_{r_n}$, where $0 < r_n \leq r'_n$. It is simple to show that, when restricted to non-constant modes, its norm is estimated in the following way.

Proposition 2.3. *If $\phi_n > 1$ and $0 < r_n \leq r'_n - \log(\phi_n)$, then $\|\mathcal{I}_n(\mathbb{I} - \mathbb{E})\| \leq \phi_n^{-1}$.*

2.10. **Elimination of non-resonant modes.** The theorem below (to be proven in Section 4.1) states the existence of a nonlinear change of coordinates isotopic to the identity that cancels the I_n^- modes of any X as in (2.5). We are eliminating only the far from resonance modes, this way avoiding the complications usually related to small divisors.

For given $\varepsilon > 0$, denote by $\mathcal{V}_\varepsilon$ the open ball in V_{r_n} with radius ε and centred at $Y_\lambda^{(n)}$. In addition, let

$$K_n = \frac{4}{2\pi} \max\left(2(1+2\pi), \frac{8\pi\rho_n + \sigma_n}{\min_{\boldsymbol{\nu}\in\mathbb{Z}^2-\{0\},\|\boldsymbol{\nu}\|\leq 4\rho_n/\sigma_n} |2\rho_n - \boldsymbol{\nu}\cdot\boldsymbol{\omega}^{(n)}|}\right), \tag{2.36}$$

$$\varepsilon_n = \frac{\sigma_n^2}{672\, K_n^2} \frac{1}{\|\boldsymbol{\omega}^{(n)}\| + \|u^{(n)}\|}. \tag{2.37}$$

Theorem 2.4. *If $X_\lambda \in \mathcal{V}_{\varepsilon_n/2}$ and $\lambda \in \mathcal{P}_{\varepsilon_n/2}$, there is an isotopic to the identity diffeomorphism*

$$\begin{aligned} \psi\colon \mathcal{D}_{r_n} \times \mathrm{SL}(2,\mathbb{C}) &\to \mathcal{D}_{r_n} \times \mathrm{SL}(2,\mathbb{C}) \\ (\boldsymbol{x},\boldsymbol{y}) &\mapsto (\boldsymbol{x}, U(\boldsymbol{x})\,\boldsymbol{y}), \end{aligned} \tag{2.38}$$

with $U \in \mathcal{A}'_{r_n}$ satisfying $\mathbb{I}_n^-\psi^*X_\lambda = 0$. This defines the maps $\mathfrak{U}_n\colon \mathcal{V}_{\varepsilon_n/2} \to \mathcal{A}'_{r_n}$, $X_\lambda \mapsto U$ and $\mathcal{U}_n\colon \mathcal{V}_{\varepsilon_n/2} \to \mathbb{I}_n^+ V_{r_n}$, $X_\lambda \mapsto \psi^*X_\lambda$ which are analytic and verify the inequalities

$$\begin{aligned} \|\mathfrak{U}_n(X_\lambda) - I\|'_{r_n} &\leq \frac{6K_n}{\sigma_n} \|\mathbb{I}_n^- X_\lambda\|_{r_n} \\ \|\mathcal{U}_n(X_\lambda) - \mathbb{E}X_\lambda\|_{r_n} &\leq 2\|(\mathbb{I}-\mathbb{E})X_\lambda\|_{r_n}, \end{aligned} \tag{2.39}$$

where I is the identity matrix. Moreover, $\mathfrak{U}_n(X_\lambda)\colon \mathbb{R}^2 \to \mathrm{SL}(2,\mathbb{R})$.

Lemma 2.5. *If u is (C,τ)-diophantine with respect to $\boldsymbol{\omega}$, then for $\boldsymbol{\nu} \in \mathbb{Z}^2$ such that $\|\boldsymbol{\nu}\| \leq 4\rho_n/\sigma_n$,*

$$|2\rho_n - \boldsymbol{\nu}\cdot\boldsymbol{\omega}^{(n)}| \geq \frac{C\sigma_n^\tau \beta_{n-1}^{2\tau-1}}{2(12\rho)^\tau}. \tag{2.40}$$

Proof. Using the fact that u is diophantine with respect to $\boldsymbol{\omega}$ in accordance with (1.7), we get

$$\begin{aligned} \min_{0<\|\boldsymbol{\nu}\|\leq 4\rho_n/\sigma_n} |2\rho_n - \boldsymbol{\nu}\cdot\boldsymbol{\omega}^{(n)}| &= \beta_{n-1}^{-1} \min_{0<\|\boldsymbol{\nu}\|\leq 4\rho_n/\sigma_n} |2\rho - {}^\top P^{(n)}\boldsymbol{\nu}\cdot\boldsymbol{\omega}| \\ &\geq \frac{C\sigma_n^\tau}{4^\tau \rho_n^\tau \beta_{n-1} \|{}^\top P^{(n)}\|^\tau}. \end{aligned} \tag{2.41}$$

Using (2.17) and the estimate for $\|{}^\top P^{(n)}\|$ obtained in section 2.3, we complete the proof. □

2.11. **Renormalization scheme.** The *nth step renormalization* operator is

$$\mathcal{R}_n = \mathcal{U}_n \circ \mathcal{I}_n \circ \mathcal{L}_n \circ \mathcal{R}_{n-1}, \tag{2.42}$$

and

$$\mathcal{R}_0(X_\lambda) = \mathcal{U}_0(X_{\Lambda_0(\lambda)}), \tag{2.43}$$

where $\Lambda_0(\lambda) = \lambda + \mathbb{E}f_\lambda$ and the resonance cones are given by the choice

$$\sigma_n = \frac{\alpha_n\beta_n\|\boldsymbol{\Omega}^{(n+1)}\|}{\|T^{(n+1)^{-1}}\|\,\|\boldsymbol{\Omega}^{(n)}\|}. \tag{2.44}$$

Notice that $\mathcal{R}_n(Y_\lambda) = Y^{(n)}_{\Lambda_n\cdots\Lambda_0(\lambda)}$. In case a vector field X is real-analytic, the same is true for $\mathcal{R}_n(X)$.

For a given non-zero vector $\boldsymbol{\omega} = (\alpha, 1)$ and $u \in \mathfrak{sl}(2, \mathbb{R})$ diophantine with respect to $\boldsymbol{\omega}$ with exponent τ, we define the sequence of analyticity radii

$$r_n = \frac{1}{B_{n-1}} \left[r_0 - \sum_{i=0}^{n-1} B_i \log(\phi_{i+1}) \right] \tag{2.45}$$

where

$$\phi_n = 2\frac{\Theta_{n-1}}{\alpha_{n-1}\Theta_n}, \qquad \Theta_n = \beta_n^{6\tau+8} \tag{2.46}$$

and $B_n = \prod_{i=0}^{n} A_i$.

Remark 2.6. Replacing (2.44) inside (2.29) we get

$$A_n = \alpha_n(1+\beta_n)\|\boldsymbol{\Omega}^{(n+1)}\| \, \|\boldsymbol{\Omega}^{(n)}\|^{-1}. \tag{2.47}$$

So, there is a constant $c > 0$ (independent of n) satisfying

$$\frac{1}{2} \le \frac{B_n}{\beta_n} \le 2 \prod_{i=0}^{n} (1+\beta_i) \le c. \tag{2.48}$$

2.12. **Brjuno frequencies.** An irrational number α is a Brjuno number if

$$\sum_{n \ge 1} \frac{\log(q_{n+1})}{q_n} < +\infty. \tag{2.49}$$

The set of all Brjuno numbers is denoted by BC (see [23] for a profound study of these numbers).

Lemma 2.7. *$\alpha \in BC$ iff*

$$\mathcal{B}(\alpha) := \sum_{n \ge 0} B_n \log(\phi_{n+1}) < +\infty. \tag{2.50}$$

Proof. Using (2.12) we get

$$B_{n-1} \log\left(\frac{1}{\beta_n}\right) \ll \frac{\log(q_{n+1})}{q_n} \ll B_{n-1} \log\left(\frac{1}{\beta_n}\right). \tag{2.51}$$

So, $\alpha \in BC$ iff $\sum_{n \ge 0} B_n \log\left(\beta_{n+1}^{-1}\right) < +\infty$. From (2.45) and the fact that the series $\sum \beta_n$, $\sum \beta_n \log(1/\alpha_n)$ and $\sum \beta_n \log(1/\beta_n)$ always converge for any irrational α, it is simple to check that

$$\sum_{n \ge 0} B_n \log\left(\beta_{n+1}^{-1}\right) \ll \sum_{n \ge 0} B_n \log\left(\phi_{n+1}^{-1}\right) \ll \sum_{n \ge 0} B_n \log\left(\beta_{n+1}^{-1}\right). \tag{2.52}$$

This proves the assertion. □

2.13. **Trivial limit of renormalization.** The convergence of the renormalization scheme now follows directly from our construction.

Theorem 2.8. *If $\alpha \in BC$, $r > \mathcal{B}(\alpha)$ and u is (C, τ)-diophantine with respect to $\boldsymbol{\omega} = (\alpha, 1)$, there exist $K, \mu > 0$, an open ball $\Delta \subset \mathfrak{a}_r$ centred at the origin, and an analytic map from Δ to $\mathcal{P}_\mu \cap \mathfrak{sl}(2, \mathbb{R})$ given by $f \mapsto a$, such that, for all $n \in \mathbb{N} \cup \{0\}$:*

1. *there exist $c_1, c_2 > 0$ such that $c_1 \le r_n \beta_{n-1} \le c_2$;*
2. *$X_\lambda = Y_\lambda + (0, f)$, with $f \in \Delta$ and $\lambda \in \mathcal{P}_\mu$, is inside the domain of $\mathcal{R}_n$ and*

$$\|\mathcal{R}_n(X_a) - \mathcal{R}_n(Y_a)\|_{r_n} \le K\Theta_n \|X_a - Y_a\|_r.$$

Proof. Let $r_0 = r > \mathcal{B}(\alpha)$. By (2.45), (2.48) and Lemma 2.7 we have $1 \ll r_n\beta_{n-1} \ll 1$ for all $n \in \mathbb{N}$. Thus (1).

Suppose Δ has radius $c > 0$. If $c \le \varepsilon_0/2$ and μ is such that $\lambda_0 = \Lambda_0(\lambda) \in \mathcal{P}_{\varepsilon_0/2}$ for $\lambda \in \mathcal{P}_\mu$, we can use Theorem 2.4 to obtain $X^{(0)}_{\lambda_0} \in \mathbb{I}^+_0 V_{r_0}$ with

$$\|\mathcal{R}_0(X_\lambda) - Y^{(0)}_{\lambda_0}\|_{r_0} \le 2\|X_{\lambda_0} - Y_{\lambda_0}\|_r.$$

By choosing $K = 2\Theta_0^{-1}$, we get

$$\|\mathcal{R}_0(X_\lambda) - Y^{(0)}_{\lambda_0}\|_{r_0} \le K\Theta_0\|X_\lambda - Y_\lambda\|_r.$$

For $n \in \mathbb{N}$ let $\mu_n = \mu\Theta_n$. Assume that $\lambda_{n-1} = \Lambda_{n-1}\dots\Lambda_0(\lambda) \in \mathcal{P}_{\mu_{n-1}}$, and suppose that $\mathcal{R}_{n-1}(X_\lambda)$ is in $\mathbb{I}^+_{n-1}V_{r_{n-1}}$ satisfying

$$\|\mathcal{R}_{n-1}(X_\lambda) - Y^{(n-1)}_{\lambda_{n-1}}\|_{r_{n-1}} \le K\Theta_{n-1}\|X_\lambda - Y_\lambda\|_r.$$

As the hypothesis yields (2.24) and $S_{n-1} < Kc\Theta_{n-1} < \mu_{n-1}/4$ for $c < \mu/(4K)$, Proposition 2.1 is valid and together with Propositions 2.2 and 2.3, they can be used to estimate:

$$\begin{aligned}\|\mathcal{I}_n \circ \mathcal{L}_n \circ \mathcal{R}_{n-1}(X_\lambda) - Y^{(n)}_{\lambda_n}\|_{r_n} &\le \alpha_{n-1}^{-1}\phi_n^{-1}K\Theta_{n-1}\|X_\lambda - Y_\lambda\|_r \\ &= \frac{1}{2}K\Theta_n\|X_\lambda - Y_\lambda\|_r,\end{aligned} \tag{2.53}$$

where we restrict λ such that $\lambda_n = \Lambda_n(\lambda_{n-1}) \in \mathcal{P}_{\mu_n}$. This vector field is inside the domain of $\mathcal{U}_n$, because $\frac{1}{2}c\,K\Theta_n \le \varepsilon_n$ and $\mu_n \le \varepsilon_n/2$ for small enough c and μ, as required in Theorem 2.4. This follows from $\varepsilon_n \gg \beta_n^{6(\tau+1)} \ge \Theta_n$ by Lemma 2.5.

We then use (2.39) to obtain the estimate

$$\|\mathcal{R}_n(X_\lambda) - Y^{(n)}_{\lambda_n}\|_{r_n} \le K\Theta_n\|X_\lambda - Y_\lambda\|_r. \tag{2.54}$$

Now, we want to show that the following limit exists,

$$a = \lim_{m\to+\infty} \Lambda_0^{-1}\cdots\Lambda_m^{-1}(0). \tag{2.55}$$

From Proposition 2.1,

$$\Lambda_n^{-1}(\cdot) = (\mathrm{Id} + g_n)(\alpha_{n-1}\cdot) \tag{2.56}$$

is a map from $\mathcal{P}_{\mu_n}$ into $\mathcal{P}_{\mu_{n-1}}$, where g_n, defined on $\mathcal{P}_{\mu_{n-1}/4}$, is given by

$$g_n = (\mathrm{Id} + F)^{-1} - \mathrm{Id}, \qquad F\colon \lambda \mapsto \mathbb{E}f^{(n-1)}_\lambda, \tag{2.57}$$

and $g_n(\mathcal{P}_{\mu_{n-1}/4})$ is contained in a fixed set. By induction,

$$\Lambda_0^{-1}\dots\Lambda_n^{-1}(0) = g_0(\xi_0) + \sum_{i=1}^{n}\beta_{i-1}g_i(\xi_i), \tag{2.58}$$

for some $\xi_k \in \mathcal{P}_{\mu_{k-1}/4}$. Thus, there exists in $\mathfrak{sl}(2,\mathbb{C})$

$$a = g_0(\xi_0) + \sum_{i=1}^{+\infty}\beta_{i-1}g_i(\xi_i), \tag{2.59}$$

unless X is real which clearly gives $a \in \mathfrak{sl}(2,\mathbb{R})$. The map $X \mapsto a$ is analytic since the convergence is uniform.

To complete the proof of (2), we now restrict to the case $\lambda = a$. From Proposition 2.1, for each $f \in \Delta$, we have the nested sequence $\Lambda_n^{-1}(\mathcal{P}_{\mu_n}) \subset \mathcal{P}_{\mu_{n-1}}$. So, as $0 \in \cap_{i\in\mathbb{N}}\mathcal{P}_{\mu_i}$, there exist

$$\Lambda_n\dots\Lambda_0(a) = \lim_{m\to+\infty}\Lambda_{n+1}^{-1}\dots\Lambda_m^{-1}(0) \in \mathcal{P}_{\mu_n}. \tag{2.60}$$

□

Remark 2.9. The above can be generalized for a small analyticity radius r by considering a sufficiently large N and applying the above theorem to $\widetilde{X} = \mathcal{U}_N\mathcal{L}_N \ldots \mathcal{U}_1\mathcal{L}_1\mathcal{U}_0(X)$, where X is close enough to Y. We recover the large strip case since r_N is of the order of β_{N-1}^{-1}. Notice that r_N is obtained by just applying (2.35) N times, because we are not cutting-off the analyticity strip. It remains to check that $r_N > \mathcal{B}(\alpha_N)$. This follows from $\mathcal{B}(\alpha_N) = B_{N-1}^{-1}[\mathcal{B}(\alpha) - \mathcal{B}_N(\alpha)]$ where $\mathcal{B}_N(\alpha)$ is the sum of the first N terms in $\mathcal{B}(\alpha)$ and $\mathcal{B}_N(\alpha) \to \mathcal{B}(\alpha)$ as N gets large.

3. Analytic (fibered) conjugacy

Theorem 2.8 allows us to construct an analytic conjugacy of the type (1.3) between the flows generated by X_a and Y_0, and to prove Theorem 1.3.

By taking $f \in \Delta$ (by convenience of notations we shall write equivalently $X_a = Y_a + (0, f) \in \Delta$) we have $\mathcal{R}_n(X_a) \in \mathbb{I}_n^+ V_{r_n}$ and

$$\begin{aligned}\mathcal{R}_n(X_a) &= \beta_{n-1}^{-1}(\psi_0 \circ L_1 \circ \psi_1 \cdots L_n \circ \psi_n)^*(X_a)\\ &= \beta_{n-1}^{-1}\chi_n^*(X_a),\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\chi_n(\boldsymbol{x}, \boldsymbol{y}) &= (P^{(n)}\boldsymbol{x}, U_0(T^{(1)^{-1}} \cdots T^{(n)^{-1}}\boldsymbol{x}) \cdots U_{n-1}(T^{(n)^{-1}}\boldsymbol{x}) U_n(\boldsymbol{x})\, \boldsymbol{y})\\ &= (P^{(n)}\boldsymbol{x}, U_0(P^{(0)}P^{(n)^{-1}}\boldsymbol{x}) \cdots U_{n-1}(P^{(n-1)}P^{(n)^{-1}}\boldsymbol{x}) U_n(\boldsymbol{x})\, \boldsymbol{y}),\end{aligned} \tag{3.2}$$

and

$$\psi_k(\boldsymbol{x}, \boldsymbol{y}) = \psi(\mathcal{I}_k\mathcal{L}_k\mathcal{R}_{k-1}(X_a))(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, U_k(\boldsymbol{x})\, \boldsymbol{y}) \tag{3.3}$$

is the map in Theorem 2.4 at the kth step. Notice that if $\mathcal{R}_n(X_a) = Y_{a_n}^{(n)}$ for $n \in \mathbb{N}$ and

$$a_n = \Lambda_n \cdots \Lambda_0(a), \tag{3.4}$$

X_a is analytically reducible to Y_0. It remains to study the case $\mathcal{R}_n(X_a) - Y_{a_n}^{(n)} \to 0$.

Given $X \in \Delta$, define the isotopic to the identity diffeomorphism

$$W_n(X)(\boldsymbol{x}) = U_n(P^{(n)}\boldsymbol{x}), \quad \boldsymbol{x} \in P^{(n)^{-1}}\mathcal{D}_{r_n}. \tag{3.5}$$

If X is real-analytic, then $W_n(X) \in C^\omega(\mathbb{R}^2, \mathrm{SL}(2, \mathbb{R}))$, since this property holds for U_n. We also have $W_n(Y_\lambda) = I$.

Take a sequence $R_n > 0$ such that $R_n\|P^{(n)}\| \le r_n$ and

$$R \le R_n \le R_{n-1} \tag{3.6}$$

for some constant $R > 0$.

Lemma 3.1. *For all $n \in \mathbb{N} \cup \{0\}$, $W_n \colon \Delta \to \mathcal{A}_{R_n}$ is analytic and satisfies*

$$\|W_n(X_a) - I\|_{R_n} \le c\Theta_n^{1/2}\|X_a - Y_a\|_r, \quad X \in \Delta, \tag{3.7}$$

with some constants $c > 0$.

Proof. For $X_a \in \Delta$, in view of (2.39), we get

$$\begin{aligned}\|W_n(X_a) - I\|_{R_n} &= \|U_n \circ P^{(n)} - I\|_{R_n}\\ &\le \frac{6K_n}{\sigma_n}\|\mathcal{I}_n\mathcal{L}_n\mathcal{R}_{n-1}(X_a) - Y_{a_n}^{(n)}\|_{r_n}\\ &\ll \frac{\beta_{n-1}^{1/2}}{\varepsilon_n^{1/2}}\Theta_n\|X_a - Y_a\|_r \ll \Theta_n^{1/2}\|X_a - Y_a\|_r.\end{aligned} \tag{3.8}$$

We can bound the above by (3.7). From the properties of $\mathfrak{U}_n$, W_n is analytic. ☐

Consider the analytic map $H_n\colon \Delta \to \mathcal{A}_{R_n}$ defined by

$$H_n(X) = W_0(X)\dots W_n(X). \tag{3.9}$$

Lemma 3.2. *There exists $c>0$ such that for $X_a \in \Delta$ and $n \in \mathbb{N}$,*

$$\|H_n(X_a) - H_{n-1}(X_a)\|_{R_n} \leq c\Theta_n^{1/2}\|X_a - Y_a\|_r. \tag{3.10}$$

Proof. We have

$$H_n(X_a) - H_{n-1}(X_a) = W_1(X_a)\dots W_{n-1}(X_a)[W_n(X_a) - I]. \tag{3.11}$$

Hence,

$$\begin{aligned}\|H_n(X_a) - H_{n-1}(X_a)\|_{R_n} &\leq \|W_1(X_a)\|_{R_1}\dots\|W_{n-1}(X_a)\|_{R_{n-1}}\|W_n(X_a) - I\|_{R_n} \\ &\leq \|W_n(X_a) - I\|_{R_n}\prod_{i=1}^{n-1}(1 + \|W_i(X_a) - I\|_{R_i}) \\ &\ll \Theta_n^{1/2}\|X_a - Y_a\|_r,\end{aligned} \tag{3.12}$$

where we have used Lemma 3.1. ☐

Lemma 3.3. *There exists an analytic map $H\colon \Delta \to \mathcal{A}_R$ such that for $X_a \in \Delta$,*

$$H(X_a) = \lim_{n\to+\infty} H_n(X_a)$$

and

$$\|H(X) - I\|_R \leq c\|X_a - Y_a\|_r, \tag{3.13}$$

for some $c>0$. If $X \in \Delta$ is real-analytic, then $H(X) \in C^\omega(\mathbb{R}^2, \mathrm{SL}(2,\mathbb{R}))$.

Proof. Lemma 3.2 implies the existence of the limit $H_n(X_a) \to H(X_a)$ as $n \to +\infty$, for each $X_a \in \Delta$, in the space $\mathcal{A}_R$. Moreover, $\|H(X_a) - I\|_R \leq c\|X_a - Y_a\|_r$. The convergence of H_n is uniform in Δ so H is analytic. The fact that, for real-analytic X, $H(X_a)$ take real values for real arguments, follows from the same property of $W_n(X_a)$. ☐

Lemma 3.4. *For every $X \in \Delta$, $\psi^*(X_a) = Y_0$ on $\mathcal{D}_R \times \mathrm{SL}(2,\mathbb{C})$, where $\psi\colon (\boldsymbol{x},\boldsymbol{y}) \mapsto (\boldsymbol{x}, H(X_a)\boldsymbol{y})$.*

Proof. For each $n \in \mathbb{N}$ the definition of $H_n(X_a)$ and (3.1) imply that

$$\widehat{H}_n(X_a)^*(X_a) = \beta_{n-1}M_n^*(\mathcal{R}_n(X_a)), \tag{3.14}$$

where $\widehat{H}_n(X_a)\colon (\boldsymbol{x},\boldsymbol{y}) \mapsto (\boldsymbol{x}, H_n(X_a)(\boldsymbol{x})\,\boldsymbol{y})$ and $M_n\colon (\boldsymbol{x},\boldsymbol{y}) \mapsto ({P^{(n)}}^{-1}\boldsymbol{x}, \boldsymbol{y})$.

Since ${P^{(n)}}^{-1}\boldsymbol{\omega}^{(n)} = \beta_{n-1}^{-1}\boldsymbol{\omega}$ and $u^{(n)} = \beta_{n-1}^{-1}u$, the r.h.s. of (3.14) can be written as

$$(\boldsymbol{\omega}, u + \beta_{n-1}[a_n + f_{a_n}^{(n)} \circ P^{(n)}]). \tag{3.15}$$

Using the fact that

$$\lim_{n\to+\infty}\sup_{\boldsymbol{x}\in\mathcal{D}_R}\beta_{n-1}\|a_n + f_{a_n}^{(n)}(P^{(n)}\boldsymbol{x})\| = 0 \tag{3.16}$$

and the convergence of H_n, we complete the proof. ☐

4. Elimination of non-resonant modes

4.1. **Homotopy method.** As n and λ are fixed, in this section we will drop these subscripts from our notations. Denote $X = (\boldsymbol{\omega}, u + \lambda + f)$.

The coordinate transformation ψ will be determined by some U in

$$\mathcal{B}_\delta = \left\{ U \in \mathbb{I}^- \mathcal{A}'_r \colon \|U - I\|'_r < \delta \right\},$$

for

$$\delta = 6K\varepsilon/\sigma < 1. \tag{4.1}$$

Define the operator

$$\begin{aligned} \mathcal{F} \colon \mathcal{B}_\delta &\to \mathbb{I}^- \mathcal{A}_r \\ U &\mapsto \mathbb{I}^-(L_{\boldsymbol{\omega}} U \cdot U^{-1} + \operatorname{Ad}_U f). \end{aligned} \tag{4.2}$$

If U is real-analytic, then $\mathcal{F}(U)$ is also real-analytic. The derivative of $\mathcal{F}$ at U is the linear map from $\mathbb{I}^- \mathcal{A}'_r$ to $\mathbb{I}^- \mathcal{A}_r$ (we use the same notations both for the base and tangent spaces) given by

$$D\mathcal{F}(U)\, H = \mathbb{I}^-(L_{\boldsymbol{\omega}} H - L_{\boldsymbol{\omega}} U \cdot U^{-1} H - \operatorname{Ad}_U f \cdot H + Hf)\, U^{-1}. \tag{4.3}$$

We want to find a solution of

$$\mathcal{F}(U_t) = (1-t)\mathcal{F}(U_0), \tag{4.4}$$

with $0 \le t \le 1$ and initial condition $U_0 = I$. Differentiating the above equation with respect to t, we get

$$D\mathcal{F}(U_t) \frac{dU_t}{dt} = -\mathcal{F}(I). \tag{4.5}$$

Proposition 4.1. *There is $\delta > 0$ such that if $U \in \mathcal{B}_\delta$, then $D\mathcal{F}(U)^{-1}$ is a bounded linear operator from $\mathbb{I}^- \mathcal{A}_r$ to $\mathbb{I}^- \mathcal{A}'_r$ and*

$$\|D\mathcal{F}(U)^{-1}\| < \delta/\varepsilon.$$

From the above proposition (to be proved in Section 4.2) we integrate (4.5) with respect to t, obtaining the integral equation:

$$U_t = I - \int_0^t D\mathcal{F}(U_s)^{-1}\, \mathcal{F}(I)\, ds. \tag{4.6}$$

In order to check that $U_t \in \mathcal{B}_\delta$ for any $0 \le t \le 1$, we estimate its norm:

$$\begin{aligned} \|U_t - I\|'_r &\le t \sup_{v \in \mathcal{B}_\delta} \|D\mathcal{F}(v)^{-1} \mathcal{F}(I)\|'_r \\ &\le t \sup_{v \in \mathcal{B}_\delta} \|D\mathcal{F}(v)^{-1}\|\, \|\mathbb{I}^- f\|_r < t\delta \|\mathbb{I}^- f\|_r/\varepsilon, \end{aligned} \tag{4.7}$$

so, $\|U_t - I\|'_r < \delta$. Therefore, the solution of (4.4) exists in $\mathcal{B}_\delta$ and is given by (4.6). Moreover, if X is real-analytic, then U_t takes real values for real arguments.

In view of

$$\mathbb{I}^+(\operatorname{Ad}_U f - u - \lambda) = \mathbb{I}^+ \left[(U - I) f (U^{-1} - I) + (U - I)\widetilde{f} + \widetilde{f}(U^{-1} - I) + \widetilde{f} \right], \tag{4.8}$$

where $\widetilde{f} = f - u - \lambda$, we get

$$\begin{aligned}\|U_t^* X - \mathbb{E}X\|_r \leq & \|\mathbb{I}^+ L_{\boldsymbol{\omega}}(U - I)\cdot(U^{-1} - I)\|_r + \|\mathbb{I}^+(\mathrm{Ad}_U f - u - \lambda)\|_r + (1-t)\|\mathbb{I}^- f\|_r \\ \leq & 2\|\boldsymbol{\omega}\|\,\|U\|_r\,\|U - I\|_r\|U - I\|'_r + 2\|U\|\,(\|u\| + \|\lambda\| + \|\widetilde{f}\|)\|U - I\|_r^2 \\ & + \|\widetilde{f}\|_r(1 + 2\|U\|_r)\|U - I\|_r + \|\widetilde{f}\|_r + (1-t)\|\mathbb{I}^- f\|_r \\ \leq & (3-t)\|\widetilde{f}\|_r.\end{aligned} \tag{4.9}$$

The result corresponds to the case $t = 1$.

4.2. **Proof of Proposition 4.1.**

Lemma 4.2. *If $g = \lambda + (\mathbb{I} - \mathbb{E})f$, we have that $D\mathcal{F}(I)^{-1}\colon \mathbb{I}^-\mathcal{A}_r \to \mathbb{I}^-\mathcal{A}'_r$ is continuous and*

$$\|D\mathcal{F}(I)^{-1}\| < \frac{1}{\sigma/K - 2\|g\|_r}. \tag{4.10}$$

Proof. From (4.3) one has

$$\begin{aligned} D\mathcal{F}(I)\,H &= \mathbb{I}^-(L_{\boldsymbol{\omega}} + \mathrm{ad}_f)\,H \\ &= \left[\mathbb{I} + \mathbb{I}^-\,\mathrm{ad}_g\,(L_{\boldsymbol{\omega}} + \mathrm{ad}_u)^{-1}\right](L_{\boldsymbol{\omega}} + \mathrm{ad}_u)\,H,\end{aligned} \tag{4.11}$$

where $\mathrm{ad}_b\, A = Ab - bA$. Thus, the inverse of this operator, if it exists, is given by

$$D\mathcal{F}(I)^{-1} = (L_{\boldsymbol{\omega}} + \mathrm{ad}_u)^{-1}\left[\mathbb{I} + \mathbb{I}^-\,\mathrm{ad}_g\,(L_{\boldsymbol{\omega}} + \mathrm{ad}_u)^{-1}\right]^{-1}. \tag{4.12}$$

By looking at the spectral properties of the operator $(2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} I + \mathrm{ad}_u)$, with the spectrum of ad_u being $\{0, \pm 4\pi\mathrm{i}\rho\}$, it is possible to write

$$(L_{\boldsymbol{\omega}} + \mathrm{ad}_u)H(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I^-} S\Lambda_{\boldsymbol{k}}S^{-1}H_{\boldsymbol{k}}\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.13}$$

where

$$\Lambda_{\boldsymbol{k}} = (2\pi\mathrm{i})\,\mathrm{diag}(\boldsymbol{k}\cdot\boldsymbol{\omega}, \boldsymbol{k}\cdot\boldsymbol{\omega}, \boldsymbol{k}\cdot\boldsymbol{\omega} + 2\rho, \boldsymbol{k}\cdot\boldsymbol{\omega} - 2\rho) \tag{4.14}$$

and

$$S = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & \mathrm{i} & -\mathrm{i} \\ -1 & 0 & \mathrm{i} & -\mathrm{i} \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{4.15}$$

So, we have the linear map from $\mathbb{I}^-\mathcal{A}_r$ to $\mathbb{I}^-\mathcal{A}'_r$,

$$(L_{\boldsymbol{\omega}} + \mathrm{ad}_u)^{-1}\,F(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I^-} S\Lambda_{\boldsymbol{k}}^{-1}S^{-1}F_{\boldsymbol{k}}\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{4.16}$$

If $\boldsymbol{k} \in I^-$ and $\|\boldsymbol{k}\| > 4\rho/\sigma$, by (2.28)

$$\frac{|\boldsymbol{k}\cdot\boldsymbol{\omega} \pm 2\rho|}{\|\boldsymbol{k}\|} \geq \frac{|\boldsymbol{k}\cdot\boldsymbol{\omega}| - 2\rho}{\|\boldsymbol{k}\|} > \frac{\sigma}{2}. \tag{4.17}$$

The number of remaining modes $\boldsymbol{k} \in I^-$ is finite.

The above inequalities and (2.28) imply that

$$\begin{aligned}\|(L_{\boldsymbol{\omega}} + \mathrm{ad}_u)^{-1}\,F\|'_r &\leq \frac{4}{2\pi}\sum_{\boldsymbol{k}\in I^-}\max\left(\frac{1 + 2\pi\|\boldsymbol{k}\|}{|\boldsymbol{k}\cdot\boldsymbol{\omega}|}, \frac{1 + 2\pi\|\boldsymbol{k}\|}{|\boldsymbol{k}\cdot\boldsymbol{\omega} \pm 2\rho|}\right)\|F_{\boldsymbol{k}}\|\mathrm{e}^{r\|\boldsymbol{k}\|} \\ &< \frac{K}{\sigma}\|F\|_r.\end{aligned} \tag{4.18}$$

Hence, $\|(L_{\boldsymbol{\omega}}+\operatorname{ad}_u)^{-1}\|<K/\sigma$. It is possible to bound from above the norm of ad_g by $2\|g\|_r$. Therefore,

$$\|\mathbb{I}^-\operatorname{ad}_g(L_{\boldsymbol{\omega}}+\operatorname{ad}_u)^{-1}\|<\frac{2K}{\sigma}\|g\|_r<1,$$

and

$$\left\|\left[\mathbb{I}+\mathbb{I}^-\operatorname{ad}_g(L_{\boldsymbol{\omega}}+\operatorname{ad}_u)^{-1}\right]^{-1}\right\|<\frac{1}{1-\frac{2K}{\sigma}\|g\|_r}.$$

The statement of the lemma is now immediate. □

As r is constant, in the following we drop it from our notations.

Lemma 4.3. *Given $U\in\mathcal{B}_\delta$, the linear operator $D\mathcal{F}(U)-D\mathcal{F}(I)$ mapping $\mathbb{I}^-\mathcal{A}'_r$ into $\mathbb{I}^-\mathcal{A}_r$, is bounded and*

$$\|D\mathcal{F}(U)-D\mathcal{F}(I)\|<2\|U\|\left[\|\boldsymbol{\omega}\|(1+2\|U\|)+2\|f\|(1+\|U\|+\|U\|^2)\right]\|U-I\|. \tag{4.19}$$

Proof. In view of (4.3), we have

$$\begin{aligned}\left[D\mathcal{F}(U)-D\mathcal{F}(I)\right]H=&\mathbb{I}^-L_{\boldsymbol{\omega}}H\cdot(U^{-1}-I)-L_{\boldsymbol{\omega}}U\cdot U^{-1}HU^{-1}\\&+Hf(U^{-1}-I)+fH-\operatorname{Ad}_U f\cdot HU^{-1}.\end{aligned} \tag{4.20}$$

It is possible to estimate the norms of the above terms by

$$\begin{aligned}\|L_{\boldsymbol{\omega}}H\cdot(U^{-1}-I)\|&\le\|\boldsymbol{\omega}\|\,\|U^{-1}-I\|\|H\|',\\ \|L_{\boldsymbol{\omega}}U\cdot U^{-1}HU^{-1}\|&\le\|\boldsymbol{\omega}\|\,\|U^{-1}\|^2\|U-I\|'\|H\|,\\ \|Hf(U^{-1}-I)\|&\le\|f\|\|U^{-1}-I\|\|H\|,\\ \|fH-\operatorname{Ad}_U f\cdot HU^{-1}\|&=\|fH(U^{-1}-I)+f(U^{-1}-I)HU^{-1}+(U^{-1}-I)fU^{-1}HU^{-1}\|\\&\le\|f\|\,(1+\|U^{-1}\|+\|U^{-1}\|^2)\|U^{-1}-I\|\|H\|.\end{aligned} \tag{4.21}$$

Finally, notice that $\|U^{-1}-I\|\le\|U^{-1}\|\,\|U-I\|\le2\|U\|\,\|U-I\|$. □

Proposition 4.1 now follows from $\|U\|<1+\delta$ and

$$\begin{aligned}\|D\mathcal{F}(U)^{-1}\|&\le\left(\|D\mathcal{F}(I)^{-1}\|^{-1}-\|D\mathcal{F}(U)-D\mathcal{F}(I)\|\right)^{-1}\\&<\left\{\sigma/K-\varepsilon-2\delta\|U\|\left[\|\boldsymbol{\omega}\|(1+2\|U\|)+2\|f\|(1+\|U\|+\|U\|^2)\right]\right\}^{-1}\\&<\left\{\sigma/K-\varepsilon-56\,\delta(\|\boldsymbol{\omega}\|+\|f\|)\right\}^{-1}.\end{aligned} \tag{4.22}$$

Therefore, for δ and ε as in (4.1) and (2.37), respectively,

$$\|D\mathcal{F}(U)^{-1}\|<\frac{\delta}{\varepsilon}. \tag{4.23}$$

Appendix A. Trivial fixed points of renormalization

Here we shall describe an adaptation of the renormalization scheme in order to obtain trivial (integrable) fixed points. That is, we want to find a way to renormalize certain vector fields in such a way as to get the same vector field.

A.1. **Constant vector fields.** We will be interested in dealing with vector fields in V_r close to

$$Y(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{\omega}, u\,y), \tag{A.1}$$

where

$$u = 2\pi\rho \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{A.2}$$

and $\boldsymbol{\omega} = (\alpha, 1)$ with a quadratic irrational number α, i.e. a solution of a quadratic equation over $\mathbb{Q}$. In this case there is a matrix $T \in \mathrm{SL}(2,\mathbb{Z})$ and $\lambda \neq 0$ such that $T\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$.

A.2. **Change of basis and time rescaling.** By performing the coordinate and time transformations $\boldsymbol{x} \mapsto T\boldsymbol{x}$ and $t \mapsto \lambda^{-1}t$, we obtain the vector field

$$\mathcal{L}(Y)(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{\omega}, \lambda^{-1}u\,\boldsymbol{y}). \tag{A.3}$$

The operator $\mathcal{L}$ is the same as in sections 2.5 and 2.8. It is compact because it can be writen as the composition of a bounded map and an inclusion (into a smaller domain) which is compact.

A.3. **Resonant rotation.** For a fixed $\boldsymbol{m} \in \mathbb{Z}^2$ consider the map $B \colon \mathcal{D}_r \to \mathrm{SO}(2,\mathbb{C})$ given by

$$B(\boldsymbol{x}) = \begin{bmatrix} \cos(2\pi\boldsymbol{m}\cdot\boldsymbol{x}) & \sin(2\pi\boldsymbol{m}\cdot\boldsymbol{x}) \\ -\sin(2\pi\boldsymbol{m}\cdot\boldsymbol{x}) & \cos(2\pi\boldsymbol{m}\cdot\boldsymbol{x}) \end{bmatrix} \tag{A.4}$$

We can thus construct the diffeomorphism $\psi(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{x}, B(\boldsymbol{x})\,\boldsymbol{y})$ and, by denoting the operator $\mathcal{B} = \psi^*$, it follows immediately from (1.4) that

$$\mathcal{B}(Y)(\boldsymbol{x},\boldsymbol{y}) = \left(\boldsymbol{\omega}, 2\pi(\boldsymbol{m}\cdot\boldsymbol{\omega}) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{y} + \mathrm{Ad}_{B(\boldsymbol{x})}\, u\cdot\boldsymbol{y}\right). \tag{A.5}$$

Notice that $\mathrm{Ad}_{B(\boldsymbol{x})}\, u = u$. The derivative of $\mathcal{B}$ at Y is given by Ad_B.

A.4. **Renormalization fixed points.** We now construct a renormalization scheme based on section 2.11, including in addition the above transformation. For each step we take the same operator consisting of $\mathcal{R} = \mathcal{B}\circ\mathcal{U}\circ\mathcal{L}$, acting on the space of vector fields whose first component is equal to a fixed $\boldsymbol{\omega}$. The domain of $\mathcal{R}$ includes all vector fields whose image under $\mathcal{L}$ is inside the domain of $\mathcal{U}$. In addition, we have $D\mathcal{U}(Y) = \mathbb{I}^+$. Here we choose to have always the same resonant cone I^+.

It is simple to check that

$$\mathcal{B}\circ\mathcal{L}(Y)(\boldsymbol{x},\boldsymbol{y}) = \left(\boldsymbol{\omega}, (\rho^{-1}\boldsymbol{m}\cdot\boldsymbol{\omega} + \lambda^{-1})\,u\,\boldsymbol{y}\right). \tag{A.6}$$

We remark that $\mathcal{U} = \mathrm{Id}$ for these cases.

The fixed points of renormalization are determined by the non-zero ρ's and quadratic irrational α's for which we can find integer solutions $\boldsymbol{m}$ to the equation

$$\boldsymbol{m}\cdot\boldsymbol{\omega} = \rho(1-\lambda^{-1}). \tag{A.7}$$

A simple example is given by $\alpha = (\sqrt{5}-1)/2$ and $\rho = 1$. In this case, $T = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\lambda = \alpha$. The solution of equation (A.7) is then $\boldsymbol{m} = (-1, 0)$.

A.5. **Hyperbolicity of fixed points.** The operator $\mathcal{R}$ is differentiable in its domain and its derivative at a trivial fixed point Y is compact and essentially given by $h \mapsto \lambda^{-1} \operatorname{Ad}_B \mathbb{I}^+ h \circ T$. The study of its spectrum shows us that the modulus of the eigenvalues are zero and $|\lambda|^{-1}$. The unstable directions correspond to perturbations to the constant matrix u in $\mathfrak{sl}(2, \mathbb{R})$.

This analysis is similar to the one appearing in [16] for the renormalization of Hamiltonian systems and in [19] for the case of non-singular flows on the torus. We should then be able to prove the following result.

Theorem A.1. *If $\boldsymbol{\omega}$ has a quadratic irrational slope, then Y is a hyperbolic fixed point of $\mathcal{R}$ with a local codimension-3 stable manifold and a local 3-dimensional unstable manifold.*

Remark A.2. The above can be easily generalized to higher dimensions $d \geq 2$ by considering vectors $\boldsymbol{\omega} \in \mathbb{R}^d$ of Koch type (for $d = 2$ they are precisely the quadratic irrationals) – see [16]. Those are vectors for which we can find $T \in \mathrm{SL}(d, \mathbb{Z})$ and $0 < |\lambda_1| < 1 < |\lambda_2| \leq \cdots \leq |\lambda_d|$ such that $T\boldsymbol{\omega} = \lambda_1 \boldsymbol{\omega}$ and every λ_i is a simple eigenvalue of T. The eigenvector λ_1 would then take the role of λ. (Notice that with respect to the notations in [16, 19], here we use the inverse matrix.)

Reducibility to Y can be proved for vector fields inside the stable manifold, following the procedure of section 3.

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Departamento de Matemática ISEG, Universidade Técnica de Lisboa, Rua do Quelhas 6, 1200-781 Lisbon, Portugal
E-mail address: jldias@iseg.utl.pt