HAMILTONIAN SUSPENSION OF PERTURBED POINCARÉ SECTIONS AND AN APPLICATION

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Abstract. We construct a Hamiltonian suspension for a given symplectomorphism which is the perturbation of a Poincaré map. This is especially useful for the conversion of perturbative results between symplectomorphisms and Hamiltonian flows in any dimension $2d$. As an application, using known properties of area-preserving maps, we prove that for any Hamiltonian defined on a symplectic 4-manifold M and any point $p \in M$, there exists a C^2-close Hamiltonian whose regular energy surface through p is either Anosov or contains a homoclinic tangency.

MSC 2000: Primary: 37J45, 37D05 ; Secondary: 37D20.
keywords: Hamiltonian vector field, Anosov flow, elliptic point, homoclinic tangency.

1. Introduction and statement of the results

Let (M, ω) be a compact C^∞ symplectic $2d$-manifold, $d \geq 2$, with a smooth boundary ∂M. Let $C^s(M)$, $2 \leq s \leq \infty$, stand for the set of C^s-Hamiltonians on M constant on each connected component of ∂M. We endow $C^s(M)$ with the C^r-Whitney topology. The return map of a Hamiltonian flow to a transversal section in an level set of the Hamiltonian is called the Poincaré map (see section 2.1). Our main result states that if we perturb the Poincaré map of a periodic orbit, there is a nearby Hamiltonian realizing the new map.

Theorem 1 (Hamiltonian suspension)**.** *Let $d \geq 2$ and $H \in C^\infty(M)$ with Poincaré map f at a periodic point p. Then, for any $\epsilon > 0$ there is $\delta > 0$ such that for any symplectomorphism $\widetilde{f}$ being δ-C^3-close to f, there is a Hamiltonian $\widetilde{H}$ ϵ-C^2-close with Poincaré map $\widetilde{f}$ at p.*

The proof of the above theorem is contained in section 3. It is based on the construction using generating functions of an isotopy between f and $\widetilde{f}$, that extends to a Hamiltonian flow. The possibility of such type of suspensions of Poincaré maps was mentioned in [7] when the manifold is the annulus (see also [6]), but without an explicit construction. An approximated suspension used in [17] by Takens is insufficient for a direct relation between the dynamics of maps and flows in many applications (as the one presented below).

Date: September 25, 2012.

1.1. **An application.** A few years ago Palis conjectured that any dynamical system can be approximated in a certain topology by a hyperbolic system without cycles, or by a system exhibiting either a homoclinic tangency or a heterodimensional cycle (cf. [14, 15]). Later, Pujals and Sambarino [16] proved this conjecture for the C^1 topology in the context of diffeomorphisms on compact surfaces. Notice that there are no heterodimensional cycles for surface diffeomorphisms.

A version for flows appeared in [1] stating that on a 3-dimensional compact manifold, a vector field can be C^1-approximated by another satisfying only one of the following phenomena:

- uniform hyperbolicity with no cycles,
- a homoclinic tangency,
- a singular cycle.

It has been further conjectured ([15, Conjecture 4]) that the last situation above can be replaced by a singular hyperbolic set (see [12] for the definition).

Related results can be obtained when restricting to conservative systems. In fact, any divergence-free vector field defined on a 3-dimensional closed manifold can be C^1-approximated in the same class by a vector field either Anosov or with a homoclinic tangency associated to a hyperbolic closed orbit [4]. This was recently generalized in [9] for a d-dimensional closed manifold, $d \geq 4$: any divergence-free vector field can be C^1-approximated by another one satisfying either one of the properties of the 3-dimensional case, or with a heterodimensional cycle. Here we address the problem of obtaining a version of [4] in the Hamiltonian context.

For each $H \in C^s(M)$ one has the Hamiltonian vector field X_H and the Hamiltonian flow φ_H^t. Consider an *energy* $e \in H(M) \subset \mathbb{R}$ and the associated φ_H^t-invariant *energy level set* $H^{-1}(e)$. An *energy surface* is a connected component of $H^{-1}(e)$. We say that it is regular if it does not contain critical points. A regular energy surface is *Anosov* if it is uniformly hyperbolic (cf. [5]). It is *far from Anosov* if it is not in the closure of Anosov regular energy surfaces. Moreover, Anosov regular energy surfaces do not contain singularities or elliptic closed orbits.

Theorem 2. *Let $d = 2$, $H \in C^2(M)$ and $p \in M$. There exists a Hamiltonian C^2-close to H whose regular energy surface through p is either Anosov or else it contains a homoclinic tangency associated to some hyperbolic closed orbit.*

Recall that the existence of homoclinic tangencies is a sufficient condition to have elliptic points (see [13, 8]). We see that it is also a necessary condition for, at least, a sufficient C^1-close vector field.

Theorem 3. *Let $d = 2$, $H \in C^2(M)$ and $p \in M$ lies in an elliptic closed orbit of H. Then, there exists a Hamiltonian C^2-close to H whose regular energy surface through p has a homoclinic tangency associated to some hyperbolic closed orbit.*

In the proof of Theorem 3 (section 4.4) we apply a mechanism introduced in [10] to create homoclinic intersections by perturbations of area-preserving maps with elliptic points (see section 4.3). We use that in our context by finding a Hamiltonian flow (through Theorem 1) that yields a Poincaré map with the same properties. Theorem 2 is then a direct consequence of Theorem 3 and of the Newhouse dichotomy (Theorem 4.1).

We remark that the result in [10] holds also for real-analytic Hamiltonians. However, the problem of suspending a real-analytic Poincaré map into a Hamiltonian flow is of a very different sort because of the lack of real-analytic bump functions, and remains an open problem. So, in the absence of such suspensions, it is required to find versions of the perturbation results directly for flows.

Newhouse in [13] showed that a C^1-generic area-preserving map is either Anosov or it has a homoclinic tangency. Furthermore, he proved that homoclinic tangencies yield elliptic orbits by a perturbation. Theorem 3 is the converse of this last step in the Hamiltonian context, while Theorem 2 is the first step but obtained using a different argument.

2. Preliminaries

2.1. Poincaré maps. Consider $H \in C^s(M)$, $s \geq 2$ or $s = \infty$, and a closed orbit $\mathcal{O}$ with least period $T > 0$ for φ_H^t. At a point $p \in \mathcal{O}$ consider a transversal $\Sigma \subset M$ to the flow, i.e. a local $(2d-1)$-submanifold for which X_H is nowhere tangencial. By choosing $e = H(p)$, define the dimension $2d-2$ symplectic submanifold

$$\Sigma_e = \Sigma \cap H^{-1}(\{e\}).$$

Thus, for any $x \in \Sigma_e$,

$$T_x H^{-1}(\{e\}) = T_x \Sigma_e \oplus \mathbb{R} X_H(x),$$

where $\mathbb{R} X_H(x)$ stands for the flow direction.

Let $U \subset M$ be some open neighbourhood of p and $V = U \cap \Sigma_e$. The *Poincaré (section) map* $f \colon V \to \Sigma_e$ is the return map of φ_H^t to Σ_e. It is given by

$$f(x) = \varphi_H^{\tau(x)}(x), \qquad x \in V,$$

where τ is the return time to Σ_e defined implicitely by the relation $\varphi_H^{\tau(x)}(x) \in \Sigma_e$ and satisfying $\tau(p) = T$. In addition, p is a fixed point of f. Notice that one needs to assume that U is a small enough neighbourhood of p. Thus, f is a C^{s-1}-symplectomorphism between V and its image. Moreover, any two Poincaré section maps of the same closed orbit are conjugate by a symplectomorphism.

2.2. Hamiltonian flowtube coordinates. Denote the coordinates in $\mathbb{R}^{2d}$ as $(x_1, \dots, x_d, y_1, \dots, y_d)$. The canonical symplectic form is given by

$$\omega_0 = \sum_{i=1}^{d} dx_i \wedge dy_i.$$

The Hamiltonian vector field of any smooth Hamiltonian H on $(\mathbb{R}^{2d}, \omega_0)$ is then

$$X_H = \mathbb{J} \nabla H,$$

where $\mathbb{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and I is the $d \times d$ identity matrix.

Consider $H_0\colon \mathbb{R}^{2d} \to \mathbb{R}$ given by $H_0 = y_d$, so that

$$X_{H_0} = \frac{\partial}{\partial x_d}.$$

Hence, the flow is $\varphi^t_{H_0} = \mathrm{id} + (0, \dots, t, 0, \dots, 0)$.

The following results provide us with the above coordinates, useful to perform local perturbations of a Hamiltonian defined on any symplectic manifold (M, ω).

Theorem 2.1 (Hamiltonian flowbox, cf. e.g. [3])**.** *Let $H \in C^s(M)$, $s \geq 2$ or $s = \infty$, and $p \in M$. If $dH(p) \neq 0$, there exists a neighborhood $U \subset M$ of p and a local C^{s-1}-symplectomorphism $g\colon (U, \omega) \to (\mathbb{R}^{2d}, \omega_0)$ such that $H = H_0 \circ g$ on U.*

By considering neighbourhoods as above taken along a piece of a trajectory, we can find a small tubular neighborhood where the flow is again straightened. This is the content of the next result.

Theorem 2.2 (Hamiltonian flowtube)**.** *Let $H \in C^s(M)$, $s \geq 2$ or $s = \infty$, and a non-closed compact self-avoiding arc of trajectory $\Gamma \subset M$. There exists a neighborhood $W \subset M$ of Γ and a local C^{s-1}-symplectomorphism $\phi\colon (W, \omega) \to (\mathbb{R}^{2d}, \omega_0)$ such that $H = H_0 \circ \phi$ on W.*

3. Hamiltonian realization of a perturbed Poincaré map

Consider a Hamiltonian flow with a closed orbit and an associated Poincaré section map in an energy surface. Our goal in this section is to find a nearby Hamiltonian exhibiting a perturbed Poincaré map (Theorem 1).

3.1. Suspension of Poincaré maps.

Let $H \in C^\infty(M)$. Consider a closed orbit $\mathcal{O}$ with least period $T > 0$, $p \in \mathcal{O}$ and $e = H(p)$. The Poincaré map is given by $f\colon V \to \Sigma_e$ as in section 2.1, having a fixed point at p.

The return time $\tau\colon V \to \mathbb{R}^+$ is close to T. So, choose $T_0, T_1 > 0$ such that $T_0 + T_1 \leq \frac{1}{2}\min\{\tau(x)\colon x \in V\}$. Take the arc of trajectory

$$\Gamma = \{\varphi^t_H(p)\colon T_0 \leq t \leq T - T_1\} \subset \mathcal{O}.$$

By Theorem 2.2, in a tubular neighbourhood $W \subset M$ of Γ we have $H = H_0 \circ \phi$. One can always compose ϕ with some symplectomorphism ψ so that $S_0, S_1 \subset \psi \circ \phi(W)$, where

$$S_0 = \{(x_1, \dots, x_d, y_1, \dots, y_d) \in \mathbb{R}^{2d}\colon x_d = y_d = 0\}$$

and $S_1 = \varphi^1_{H_0}(S_0)$. We assume that ϕ is in fact $\psi \circ \phi$ in order to simplify notations. Furthermore,

$$\varphi^1_{H_0}|S_0 = \phi \circ \varphi_H^{-T_1} \circ f \circ \varphi_H^{-T_0} \circ \phi^{-1},$$

which is simply given by $\varphi^1_{H_0}(x, 0, y, 0) = (x, 1, y, 0)$ with

$$(x, y) = (x_1, \dots, x_{d-1}, y_1, \dots, y_{d-1}) \in \mathbb{R}^{2d-2}.$$

This means that $\Pi \circ \varphi^1_{H_0}|S_0 = \mathrm{id}$ by using the projection $\Pi\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d-2}$, $(x, x_d, y, y_d) \mapsto (x, y)$.

Given a C^∞-symplectomorphism $\widetilde{f}$ on V that is C^1-close to f, we want to find a Hamiltonian $\widetilde{H}$ having $\widetilde{f}$ as Poincaré map. The perturbation is constructed inside W, hence being enough to find $\widetilde{H}_0 = \widetilde{H} \circ \phi^{-1}$ such that

$$\varphi^1_{\widetilde{H}_0}|S_0 = \phi \circ \varphi_H^{-T_1} \circ \widetilde{f} \circ \varphi_H^{-T_0} \circ \phi^{-1}.$$

Then, $g = \Pi \circ \varphi^1_{\widetilde{H}_0}|S_0$ is a C^∞-symplectomorphism on $\mathbb{R}^{2d-2}$. From the above considerations we know that for any $r \geq 0$,

$$\|g - \mathrm{id}\|_{C^r} \leq c_r \|\widetilde{f} - f\|_{C^r}$$

for some $c_r > 0$ depending on H.

Let $\rho > 0$ and the euclidean open ball

$$B_\rho = \{(x,y) \in \mathbb{R}^{2d-2} \colon \|(x,y)\| < \rho\}.$$

The radius ρ is chosen small enough so that $B_\rho \times \{0 \leq x_d \leq 1, |y_d| < \rho\} \subset \phi(W)$.

Proposition 3.1. *There is $\delta, c > 0$ such that for any C^∞-symplectomorphism g compactly supported in B_ρ, δ-C^1-close to the identity, we can find $\widetilde{H}_0 \in C^\infty(\mathbb{R}^{2d})$ compactly supported in B_ρ verifying*

$$\Pi \circ \varphi^1_{\widetilde{H}_0}|S_0 = g$$

and

$$\left\|\widetilde{H}_0 - H_0\right\|_{C^2} \leq c(1 + \rho + \rho^{-1} + \rho\|g - \mathrm{id}\,\|^2_{C^3})\ \|g - \mathrm{id}\|_{C^1}. \tag{1}$$

Moreover, if g fixes the origin, then $\varphi^1_{\widetilde{H}_0}(0) = (0,1,0,0)$.

We now use the above proposition (to be proved in section 3.2 below) to complete the proof of Theorem 1. Consider

$$\widetilde{H} = \begin{cases} H, & \text{on } M \setminus W \\ H + (\widetilde{H}_0 - H_0) \circ \phi, & \text{otherwise.} \end{cases}$$

Therefore, combining the estimates above and assuming that $\widetilde{f}$ is C^3-close to f, one gets

$$\|\widetilde{H} - H\|_{C^2} \leq c\|\widetilde{f} - f\|_{C^1}$$

for some $c > 0$.

3.2. **Proof of Proposition 3.1.** Since the group of smooth symplectomorphisms isotopic to the identity is path-connected, we can always find an isotopy g_α, $\alpha \in [0,1]$, of symplectomorphisms from the identity to g. The corresponding non-autonomous vector field $X_\alpha = \dot{g}_\alpha \circ g_\alpha^{-1}$ is symplectic (for each α), and in fact Hamiltonian since we are in a simply connected space. The proof of Proposition 3.1 relies on this well-known fact, but it also requires a control on the size of the derivatives of $(x,y,\alpha) \mapsto g_\alpha(x,y)$. For this reason we need to construct g_α through a simple isotopy of generating functions, whose norms are easily estimated. Later, by adding a flow direction coordinate ($\alpha = x_d$) and its symplectic conjugate (the "energy" y_d), we will extend our Hamiltonian to $\mathbb{R}^{2d}$.

For functions $F\colon D \to \mathbb{R}^m$, $D \subset \mathbb{R}^{2d}$, consider the C^s-norm, with $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\|F\|_{C^s} = \max_{i=1,\dots,m} \max_{|\sigma|\leq s} \sup_D \left| \frac{\partial^{|\sigma|} F_i}{\partial^{\sigma_1} x_1 \dots \partial^{\sigma_{2d}} y_d} \right|$$

where $\sigma = (\sigma_1, \dots, \sigma_{2d}) \in \mathbb{N}_0^{2d}$ and $|\sigma| = \sum_i \sigma_i$. Moreover, $\langle \cdot, \cdot \rangle$ denotes the usual euclidean scalar product and we introduce the projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

Let $V \in C^\infty(\mathbb{R}^{2d-2})$ such that

$$W(x', y) = \langle x', y \rangle + V(x', y)$$

is a generating function of g. More specifically, writing $(x', y') = g(x, y)$, since $\det D_1 x' \neq 0$,

$$x = \frac{\partial W}{\partial y}(x', y) \quad \text{and} \quad y' = \frac{\partial W}{\partial x'}(x', y).$$

Therefore,

$$g(x, y) = (x, y) - \mathbb{J}\nabla V \circ G(x, y).$$

where $G(x, y) = (\pi_1 g(x, y), y)$ and $\|\nabla V\|_{C^0} = \|g - \mathrm{id}\|_{C^0}$. We assume that g is sufficiently C^1-close to the identity, thus G is a diffeomorphism.

Lemma 3.2. *For $r \geq 1$, there is $c_r > 0$ such that*

$$\|\nabla V\|_{C^r} \leq c_r \max\{1, \|G^{-1}\|_{C^r}^r\} \, \|g - \mathrm{id}\|_{C^r}.$$

Proof. Write $\phi = g - \mathrm{id}$ and $\beta = G^{-1}$ so that $\phi \circ \beta = -\mathbb{J}\nabla V$. Recall the Faà di Bruno formula for the higher derivative chain rule:

$$D^r(\phi \circ \beta) = \sum \frac{r!}{k_1! \dots k_r! 1!^{k_1} \dots r!^{k_r}} D^{|k|}\phi(\beta) \, (\underbrace{D\beta, \dots, D\beta}_{k_1}, \dots, \underbrace{D^r\beta, \dots, D^r\beta}_{k_r}) \tag{2}$$

where the sum is over every $k = (k_1, \dots, k_r) \in \mathbb{N}_0^r$ such that

$$\langle k, (1, 2, \dots, r) \rangle = r.$$

Therefore, there is a constant $c_r > 0$ depending on r, satisfying

$$\|\nabla V\|_{C^r} \leq c_r \, \max\{1, \|\beta\|_{C^r}^r\} \, \|\phi\|_{C^r},$$

where we have used that $\|\beta\|_{C^{k_i}}^{k_i} \leq \|\beta\|_{C^r}^{k_i} \leq \max\{1, \|\beta\|_{C^r}^r\}$. □

Let $\ell \in C^\infty(\mathbb{R})$ be a bump function verifying

$$\ell(\alpha) = \begin{cases} 1, & \alpha \geq \xi \\ 0, & \alpha \leq 0 \end{cases}$$

for some choice of $0 < \xi < 1$ such that $\ell' > 0$ in $(0, \xi)$. We can now construct the following smooth 1-family of generating functions:

$$W_\alpha(x', y) = \langle x', y \rangle + \ell(\alpha) \, V(x', y).$$

For each $\alpha \in \mathbb{R}$ we obtain a C^∞-symplectomorphism g_α generated by W_α. Clearly, $g_0 = \mathrm{id}$ and $g_1 = g$. Hence, g_α is a C^∞-isotopy between id and g implicitly given by

$$g_\alpha = \mathrm{id} - \ell(\alpha) \, \mathbb{J}\nabla V \circ G_\alpha,$$

where $G_\alpha = (\pi_1 g_\alpha, \pi_2)$ and $\|g_\alpha - \mathrm{id}\|_{C^0} \leq \|\nabla V\|_{C^0} = \|g - \mathrm{id}\|_{C^0}$.

Lemma 3.3. *For $r \geq 1$, there is $c_r > 0$ such that for any $\alpha \in \mathbb{R}$, if $\|g - \mathrm{id}\|_{C^1}$ is sufficiently small, then*

$$\|g_\alpha - \mathrm{id}\|_{C^r} \leq \frac{c_r}{1 - \|\nabla V\|_{C^1}} \|g - \mathrm{id}\|^r_{C^{r-1}} \|\nabla V\|_{C^r}.$$

Proof. Write $v_\alpha = -\ell(\alpha)\,\mathbb{J}\nabla V$ so that $\|v_\alpha\|_{C^r} \leq \|\nabla V\|_{C^r}$. Using again the Faà di Bruno formula,

$$D^r(g_\alpha - \mathrm{id}) = \sum_{k_r=0} c_{k,r} D^{|k|} v_\alpha(G_\alpha)\,(\underbrace{DG_\alpha, \dots, DG_\alpha}_{k_1}, \dots, \underbrace{D^{r-1}G_\alpha, \dots, D^{r-1}G_\alpha}_{k_{r-1}}) \\ + Dv_\alpha(G_\alpha)\, D^r G_\alpha,$$

where $c_{k,r}$ are the coefficients as in (2) and we have split the sum in the terms corresponding to the vectors $k = (k_1, \dots, k_{r-1}, 0)$ and $k = (0, \dots, 0, 1)$. Taking the norms, with $c_r > 0$ depending on r,

$$\|g_\alpha - \mathrm{id}\|_{C^r} \leq c_r \|v_\alpha\|_{C^r} \|g_\alpha - \mathrm{id}\|^r_{C^{r-1}} + \|v_\alpha\|_{C^1}\, \|g_\alpha - \mathrm{id}\|_{C^r}.$$

Therefore,

$$\|g_\alpha - \mathrm{id}\|_{C^r} \leq \frac{c_r}{1 - \|v_\alpha\|_{C^1}} \|g_\alpha - \mathrm{id}\|^r_{C^{r-1}} \|v_\alpha\|_{C^r}.$$

The claim follows from applying Lemma 3.2. □

Consider now the C^∞-vector field $\dot{g}_\alpha = \frac{d}{d\alpha} g_\alpha$ on $\mathbb{R}^{2d-2}$ that generates the isotopy g_α. The non-autonomous vector field

$$X_\alpha = \dot{g}_\alpha \circ g_\alpha^{-1}$$

is symplectic, i.e. $\iota_{X_\alpha}\omega_0$ is a closed 1-form. By the Poincaré lemma, since our space is simply-connected, it is also exact. Therefore, for each α there exists a C^∞-function $K_\alpha \colon \mathbb{R}^{2d-2} \to \mathbb{R}$ with compact support such that $\iota_{X_\alpha}\omega_0 = dK_\alpha$, i.e. $\nabla K_\alpha = -\mathbb{J}X_\alpha$ and using the notation of a Hamiltonian vector field

$$X_{K_\alpha} = X_\alpha.$$

Up to a constant (chosen so that K_α has compact support), it is given by

$$K_\alpha(x, y) = \int_{[0,(x,y)]} \iota_{X_\alpha}\omega_0 = \int_0^1 \langle X_{K_\alpha}(s(x, y)), (y, -x)\rangle\, ds, \tag{3}$$

where the integration is along the straight path $[0, (x, y)]$ that connects (x, y) to the origin. Notice that the vector field that determines g as the time-1 map is non-autonomous, not preserving the "energy" K. Also, $K_\alpha = 0$ for any $\alpha \notin (0, 1)$.

We can extend the dimension of the space to $\mathbb{R}^{2d}$ by considering the variables $x_d = \alpha$ (seen as the time direction) and y_d (the "energy" K).

Let $\widetilde{\ell} \in C^\infty(\mathbb{R})$ be another bump function satisfying

$$\widetilde{\ell}(y_d) = \begin{cases} 1, & |y_d| \leq \nu\rho \\ 0, & |y_d| \geq \rho \end{cases}$$

for any choice of $0 < \nu < 1$, such that $\|\widetilde{\ell}\|_{C^0} \leq 1$,

$$\|\widetilde{\ell}'\|_{C^0} \leq \frac{2}{(1-\nu)\rho} \quad \text{and} \quad \|\widetilde{\ell}''\|_{C^0} \leq \frac{4}{(1-\nu)\rho^2}.$$

We define the (autonomous) C^∞-Hamiltonian $\widetilde{H}_0\colon \mathbb{R}^{2d}\to\mathbb{R}$ as

$$\widetilde{H}_0(x,x_d,y,y_d)=H_0(y_d)+K_{x_d}(x,y)\,\widetilde{\ell}(y_d)$$

with $H_0(y_d)=y_d$. Hence,

$$\nabla(\widetilde{H}_0-H_0)=\left(\widetilde{\ell}\,\frac{\partial K}{\partial x},\widetilde{\ell}\,\frac{\partial K}{\partial x_d},\widetilde{\ell}\,\frac{\partial K}{\partial y},\widetilde{\ell}'\,K\right). \tag{4}$$

Notice that outside $\{x_d\in(0,1),|y_d|<\rho\}\subset\mathbb{R}^{2d}$ we have $\widetilde{H}_0=H_0$. By contrast, the Hamiltonian vector field for $x_d\in[0,1]$ and $|y_d|\le\nu\rho$ is

$$X_{\widetilde{H}_0}=\left(\pi_1X_K,1,\pi_2X_K,-\frac{\partial K}{\partial x_d}\right).$$

Lemma 3.4. *There is $\delta>0$ and $c>0$ such that, if $\|g-\mathrm{id}\,\|_{C^1}\le\delta$, then* (1) *holds.*

Proof. We write a dot to represent the derivative with respect to x_d and D for the derivative with respect to (x,y). Recall that $X_K(x,x_d,y,y_d)=\dot g_{x_d}\circ g_{x_d}^{-1}(x,y)$. We will use Lemmas 3.2 and 3.3 without explicit mention.

From (4) we have

$$\left\|\widetilde{H}_0-H_0\right\|_{C^1}\le\max\left\{\|K\|_{C^0},\|X_K\|_{C^0},\|\dot K\|_{C^0},\|\widetilde{\ell}'\|_{C^0}\,\|K\|_{C^0}\right\}.$$

Now, the second order derivatives of $\widetilde{H}_0$ are

$$\begin{aligned}\frac{\partial^2\widetilde{H}_0}{\partial z_i\partial z_j}&=\widetilde{\ell}\,\frac{\partial^2K}{\partial z_i\partial z_j}\\ \frac{\partial^2\widetilde{H}_0}{\partial z_i\partial x_d}&=\widetilde{\ell}\,\frac{\partial\dot K}{\partial z_i}\\ \frac{\partial^2\widetilde{H}_0}{\partial^2x_d}&=\widetilde{\ell}\,\ddot K\\ \frac{\partial^2\widetilde{H}_0}{\partial z_i\partial y_d}&=\widetilde{\ell}'\,\frac{\partial K}{\partial z_i}\\ \frac{\partial^2\widetilde{H}_0}{\partial x_d\partial y_d}&=\widetilde{\ell}'\,\dot K\\ \frac{\partial^2\widetilde{H}_0}{\partial^2y_d}&=\widetilde{\ell}''\,K\end{aligned}$$

where $z=(x,y)$ and $i,j=1,\dots,2d-2$. So,

$$\begin{aligned}\left\|\widetilde{H}_0-H_0\right\|_{C^2}\le\max\{\,&\|X_K\|_{C^1},\|\widetilde{\ell}'\|_{C^0}\,\|X_K\|_{C^0},\|\ddot K\|_{C^0},\\ &\max\{1,\|\widetilde{\ell}'\|_{C^0}\}\|\dot K\|_{C^0},\\ &\max\{1,\|\widetilde{\ell}'\|_{C^0},\|\widetilde{\ell}''\|_{C^0}\}\,\|K\|_{C^0}\Big\}.\end{aligned}$$

By writing $v=-\mathbb{J}\nabla V$, we have that

$$\|\dot g\|_{C^0}\le\|\ell\|_{C^1}\,\|v\|_{C^0}+\|v\|_{C^1}\|\dot g\|_{C^0}.$$

Therefore,

$$\|\dot g\|_{C^0}\le\frac{\|\ell\|_{C^1}\,\|g-\mathrm{id}\,\|_{C^0}}{1-\|v\|_{C^1}}\le c\,\|g-\mathrm{id}\|_{C^0}$$

for some $c > 0$. Similarly,

$$\begin{aligned}\|\ddot{g}\|_{C^0} &\leq \frac{\|\ell\|_{C^2}\,\|v\|_{C^0} + 2\,\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|\dot{g}\|_{C^0} + \|v\|_{C^2}\,\|\dot{g}\|^2_{C^0}}{1 - \|v\|_{C^1}} \\ &\leq c\,\|g - \mathrm{id}\|_{C^0}\end{aligned}$$

for some $c > 0$. Moreover,

$$\|D\dot{g}\|_{C^0} \leq \|\ell\|_{C^1}\,\|v\|_{C^1}\,\|g\|_{C^1} + \|v\|_{C^2}\,\|g\|_{C^1}\,\|\dot{g}\|_{C^0} + \|v\|_{C^1}\,\|D\dot{g}\|_{C^0},$$

thus

$$\begin{aligned}\|D\dot{g}\|_{C^0} &\leq \frac{\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|g\|_{C^1} + \|v\|_{C^2}\|\dot{g}\|_{C^0}\,\|g\|_{C^1}}{1 - \|v\|_{C^1}} \\ &\leq c\,\|g - \mathrm{id}\|_{C^1}\end{aligned}$$

for some $c > 0$.

From $\dot{X}_K = \ddot{g} \circ g^{-1} + D\dot{g} \circ g^{-1}\,\dot{g}^{-1}$ and $DX_K = D\dot{g} \circ g^{-1}\,Dg^{-1}$,

$$\|X_K\|_{C^1} \leq c\,\|g - \mathrm{id}\|_{C^1}\,.$$

From (3), $\|K\|_{C^0} \leq \rho\,\|X_K\|_{C^0}$, $\|\dot{K}\|_{C^0} \leq \rho\|X_K\|_{C^1}$ and also $\|\ddot{K}\|_{C^0} \leq \rho\|\ddot{X}_K\|_{C^0}$. Thus, it remains to bound $\|\ddot{X}_K\|_{C^0}$.

As before, we obtain the following bounds:

$$\begin{aligned}\|\dddot{g}\|_{C^0} \leq \frac{1}{1 - \|v\|_{C^1}}\Big(&\|\ell\|_{C^3}\,\|v\|_{C^0} + 3\,\|\ell\|_{C^2}\,\|v\|_{C^1}\,\|\dot{g}\|_{C^0} \\ &+ 3\,\|\ell\|_{C^1}\,\|v\|_{C^2}\,\|\dot{g}\|^2_{C^0} \\ &+ 3\,\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|\ddot{g}\|_{C^0} + \|v\|_{C^3}\,\|\dot{g}\|^3_{C^0}\Big) \\ \|D^2\dot{g}\|_{C^0} \leq \frac{1}{1 - \|v\|_{C^1}}\Big(&\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|g\|^2_{C^1} + \|\ell\|_{C^1}\,\|v\|_{C^1}\,\|D^2 g\|_{C^0} \\ &+ \|v\|_{C^3}\,\|g\|^2_{C^1}\|\dot{g}\|_{C^0} + \|v\|_{C^2}\,\|D^2 g\|_{C^0}\|\dot{g}\|_{C^0} \\ &+ 2\,\|v\|_{C^2}\,\|g\|_{C^1}\,\|\dot{g}\|_{C^1}\Big) \\ \|D\ddot{g}\|_{C^0} \leq \frac{1}{1 - \|v\|_{C^1}}\Big(&\|\ell\|_{C^2}\,\|v\|_{C^1}\,\|g\|_{C^1} + 2\,\|\ell\|_{C^1}\,\|v\|_{C^2}\,\|g\|_{C^1}\,\|\dot{g}\|_{C^0} \\ &+ 2\,\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|\dot{g}\|_{C^1} + \|v\|_{C^3}\,\|g\|_{C^1}\|\dot{g}\|^2_{C^0} \\ &+ 2\,\|v\|_{C^2}\,\|\dot{g}\|_{C^1}\|\dot{g}\|_{C^0} + \|v\|_{C^2}\,\|g\|_{C^1}\,\|\ddot{g}\|_{C^0}\Big)\end{aligned}$$

Finally, we use the fact that $\ddot{X}_K = \dddot{g} \circ g^{-1} + 2D\ddot{g} \circ g^{-1}\,\dot{g}^{-1} + D^2\dot{g} \circ g^{-1}\,(\dot{g}^{-1}, \dot{g}^{-1}) + D\dot{g} \circ g^{-1}\,\ddot{g}^{-1}$. So,

$$\|\ddot{X}_K\|_{C^0} \leq c\,\left(1 + \|g - \mathrm{id}\,\|^2_{C^3}\right)\|g - \mathrm{id}\|_{C^1}$$

for some constant $c > 0$. Evaluating all the above estimates together, one gets

$$\left\|\widetilde{H}_0 - H_0\right\|_{C^2} \leq c\,\left(1 + \rho + \rho^{-1} + \rho\|g - \mathrm{id}\,\|^2_{C^3}\right)\|g - \mathrm{id}\|_{C^1}$$

for some universal constant $c > 0$ that only depends on the norms of the bump functions. □

Remark 3.1. *In the above lemma there is the need to bound the size of higher derivatives of g. This loss of differentiability is caused by our specific construction of the isotopy g_α. It should be possible to use a different isotopy that avoids this phenomenon. Our choice was done for the sake of simplicity.*

The Hamiltonian flow for $x_d \in [0,1]$ and $|y_d| \leq \nu\rho$ is given by

$$\begin{aligned}\varphi^t_{\widetilde{H}_0}(x, x_d, y, y_d) = \big(&\pi_1 g_{x_d+t} \circ g^{-1}_{x_d}(x,y),\\ &x_d + t,\\ &\pi_2 g_{x_d+t} \circ g^{-1}_{x_d}(x,y),\\ &y_d - \int_0^t \frac{\partial K_{x_d+s}}{\partial x_d} \circ g_{x_d+t} \circ g^{-1}_{x_d}(x,y)\, ds\bigg).\end{aligned}$$

Using estimates in the proof of Lemma 3.4, one gets that the increment in the last coordinate for $t \in [0,1]$ is bounded from above by

$$\left\| \frac{\partial K}{\partial x_d} \right\|_{C^0} \leq \rho \|X_K\|_{C^0} \leq \nu\rho$$

as long as $\|g - \mathrm{id}\|_{C^1}$ is small. Finally, the time-1 flow acts on the transversal $\{(x,0,y,0)\}$ by

$$\varphi^1_{\widetilde{H}_0}(x,0,y,0) = \left(\pi_1 g(x,y), 1, \pi_2 g(x,y), -\int_0^1 \frac{\partial K_s}{\partial x_d} \circ g(x,y)\, ds\right).$$

In particular, if $g(0) = (0)$, $\varphi^1_{\widetilde{H}_0}(0) = (0,1,0,0)$ because $\frac{\partial}{\partial x_d} K(0,0) = 0$.

4. Elliptic closed orbits and homoclinic tangencies

4.1. **Homoclinic tangencies.** Take $H \in C^2(M)$, a non-constant hyperbolic closed orbit $\mathcal{O}$ and a transversal section at a point $p \in \mathcal{O}$. Let W^s_p be the stable manifold at p of the Poincaré map, and W^u_p the unstable manifold. We say that $\mathcal{O}$ has a *homoclinic tangency* at $q \neq p$ if the invariant manifolds W^s_p and W^u_p have a non transversal intersection, i.e.:

- $T_q W^s_p \cap T_q W^u_p$ contains a nonzero vector,
- $T_q W^s_p \oplus T_q W^u_p \oplus \mathbb{R}X(q) \neq T_q H^{-1}(p)$.

4.2. **Density of elliptic closed orbits.** The next result is the Hamiltonian version of the Newhouse dichotomy [13] for 4-dimensional Hamiltonians. As previously mentioned, it will be used in the proof of Theorem 2 (see section 4.4).

Theorem 4.1 ([2]). *Let $d = 2$. Given an open set $U \subset M$ intersecting a far from Anosov regular energy surface of $H \in C^2(M)$, there is a C^2-nearby Hamiltonian having an elliptic closed orbit through U. Moreover, this implies that, for far from Anosov regular energy surfaces of a C^2-generic Hamiltonian, the elliptic closed orbits are dense.*

4.3. **Creation of homoclinic tangencies.** The next result is central to the proof of Theorem 2. It deals with symplectomorphisms on a symplectic 2-manifold, i.e. area-preserving maps.

Theorem 4.2 (Gelfreich and Turaev [10]). *Let $r \in \mathbb{N} \cup \{\infty, \omega\}$. Any C^r-area-preserving map with an elliptic point can be C^r-approximated by another area-preserving map with a homoclinic tangency.*

4.4. **Proof of Theorems 2 and 3.** The proof of Theorem 3 follows from the following steps:

(1) Since elliptic closed orbits are stable, we can find a C^∞ approximation $\widetilde{H}$ keeping the same (i.e. its analytic continuation) elliptic closed orbit.
(2) Consider the C^∞ Poincaré map f of $\varphi^t_{\widetilde{H}}$ on a transversal to the elliptic closed orbit restricted to an energy surface.
(3) Use Theorem 4.2 to obtain a C^∞-symplectomorphim $\widetilde{f}$ close to f with a homoclinic tangency.
(4) Finally, Theorem 1 allows us to construct a Hamiltonian C^2-close to $\widetilde{H}$, which realizes the Poincaré map $\widetilde{f}$ on the energy surface.

Assume that the energy level $H^{-1}(\{H(p)\})$ is far from Anosov. The proof of Theorem 2 follows from Theorem 3 after applying Theorem 4.1 that gives elliptic closed orbits for some Hamiltonian C^2-close.

Finally, we would like to mention a possible alternative strategy to prove Theorem 3 without the use of Theorem 4.2. We first observe that an area-preserving diffeomorphism yielding an irrational invariant curve can be perturbed in order to create homoclinic tangencies, as proved in [11]. So, starting from a Hamiltonian with an elliptic closed orbit, one can perturb its tangent map and get a new Hamiltonian (using a version of Franks Lemma [18]) whose Poincaré map is an area-preserving map satisfying a twist condition along a diophantine invariant curve. KAM theory then assures us the stability of this structure, and a suspension of the result in [11] holds homoclinic tangencies for a nearby Hamiltonian.

Acknowledgements

The authors would like to thank Carlos Matheus for fruitful conversations and suggestions. MB was partially supported by National Funds through Fundação para a Ciência e a Tecnologia, project PEst-OE/MAT/UI0212/2011. JLD was partially supported by Fundação para a Ciência e a Tecnologia through the project “Randomness in Deterministic Dynamical Systems and Applications” PTDC/MAT/105448/2008.

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