

ON THE ATTRACTOR OF PIECEWISE EXPANDING MAPS OF THE INTERVAL

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ABSTRACT. We consider piecewise expanding maps of the interval with finitely many branches of monotonicity and show that they are generically combinatorially stable, i.e., the number of ergodic attractors and their corresponding mixing periods do not change under small perturbations of the map. Our methods provide a topological description of the attractor and, in particular, give an elementary proof of the density of periodic orbits.

1. INTRODUCTION

In this paper we study the dynamics of piecewise expanding maps of the interval. We fix $m \in \mathbb{N}$. A map $f: [0, 1] \rightarrow [0, 1]$ is called *piecewise expanding* on m intervals if there exist a constant $\sigma > 1$ and intervals I_1, \dots, I_m such that

- (1) $[0, 1] = \bigcup_{i=1}^m I_i$ and $\text{int}(I_i) \cap \text{int}(I_j) = \emptyset$ for $i \neq j$,
- (2) f is C^1 and $|f'| \geq \sigma$ on each I_i ,
- (3) f' is Lipschitz on each I_i ¹.

The dynamics of this class of maps has been widely studied as it finds applications in other areas of mathematics and in many other branches of science [3]. The theory of piecewise expanding maps is by now rather satisfactory from a probabilistic point of view. Computer simulations show that typical orbits of piecewise expanding maps display chaotic behaviour as they approach an attractor. A way to describe the chaotic behaviour on the attractor is through the study of invariant measures [13, 12]. It is well-known that piecewise expanding maps admit absolutely continuous (with respect to Lebesgue measure) invariant probability measures, known as acip's [10]. They are physically meaningful since it allows us to understand the statistical behaviour of positive Lebesgue measure sets of orbits.

Deterministic and random perturbations of piecewise expanding maps have been considered by many people, e.g., [8, 11, 1, 2, 9, 6, 4]. A key concept is stability. Roughly speaking, a map is called stable if its statistical properties are robust under small perturbations of the map. In the context of piecewise expanding maps, a map f with an acip μ

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¹This implies that $f'|_{\text{int}(I_i)}$ has a continuous extension to the closure of I_i .

is acip-stable if given any small perturbation f_ε of f we have that μ_ε converges to μ in the weak*-sense as $\varepsilon \rightarrow 0$ where μ_ε is an acip of f_ε [4].

In this paper we are interested in determining which piecewise expanding maps have robust combinatorics at the level of the attractor. To be more precise, we say that a piecewise expanding map f is *combinatorially stable* if the number of ergodic acip's of f and the corresponding mixing periods do not change in a neighbourhood of f . By mixing period we mean the number of mixing components of an ergodic acip (see (2.1)).

The following theorem is the main result of this paper.

Theorem 1.1. *Generic piecewise expanding maps on m intervals are combinatorially stable and the supports of their acips vary continuously with the map.*

The sufficient conditions on the maps for the main theorem to hold are given in Definition 3.6. They are generic by Proposition 3.9. In Section 4 we give a proof and a precise formulation of Theorem 1.1 (see Theorem 4.1).

In addition, we prove several results for piecewise expanding maps used in the proof of Theorem 1.1. These results may be of independent interest. For example, in Section 3, using rather elementary methods, we prove that periodic points of piecewise expanding maps are dense in the support of their acips. The density of periodic points for piecewise expanding maps may not be surprising for experts. However, we could not find any reference in the literature.

The strategy for proving Theorem 1.1 is the following. To any generic piecewise expanding map f and ergodic acip μ of f we associate a trapping region $U_\mu(g)$ for small perturbations g of f . This trapping region contains the support of an ergodic acip μ_g of the perturbed map g . Using the density of periodic points in the support of the acips we show that μ_g is unique, i.e., no other ergodic acip of g has its support inside $U_\mu(g)$. So we have a well defined map $\Theta_g: \mu \mapsto \mu_g$ from the set of ergodic acips of f to the set of ergodic acips of g . Then we prove that Θ_g is a bijection. This shows that f and g have the same number of ergodic acips. Working in a similar way, we conclude that f and g have the same number of mixing components.

In addition to determining the number of ergodic components, the method for proving Theorem 1.1 provides a topological description and the continuity of the immediate basins of attraction, which complements the results obtained by a spectral approach [8, 9].

Our motivation to study the combinatorial stability of piecewise expanding maps comes from a special class of two-dimensional hyperbolic maps with singularities, the strongly dissipative polygonal billiards [5]. We believe that the proof of Theorem 1.1 might be adapted to study the

ergodic properties of two-dimensional attractors (supporting SRB measures) which are close, in an appropriate sense, to a one-dimensional attractor of a piecewise expanding map. The combinatorial stability for this class of two-dimensional maps will be treated in a separate paper.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and recall a well-known theorem concerning the existence of acips for piecewise expanding maps. Several topological properties of the attractor are proved in Section 3. In Section 4, we prove our main result regarding the combinatorial stability of piecewise expanding maps.

2. PRELIMINARIES

Let f be a piecewise expanding map. Throughout the paper, we use the standard abbreviation *acip* for an invariant probability measure of f that is absolutely continuous with respect to the Lebesgue measure of $[0, 1]$. We also write ‘(mod 0)’ to specify that an equality holds up to a set of zero Lebesgue measure. The length of an interval $I \subseteq [0, 1]$ is denoted by $|I|$. Given any subset $A \subset [0, 1]$, its boundary ∂A and interior $\text{int}(A)$ are taken relative to \mathbb{R} .

2.1. Existence of acips. Given a Borel measure μ , we denote by $\text{supp } \mu$ the smallest closed set of full μ -measure.

We say that the pair (f, μ) , where μ is an acip of f , is *exact*² if

$$\bigcap_{n=0}^{\infty} f^{-n}(\mathcal{B})$$

consists of μ -null sets and its complements. Here, \mathcal{B} denotes the Borel σ -algebra.

Theorem 2.1. *Let f be a piecewise expanding map. Then,*

- (1) *there exists $1 \leq k \leq m$ such that f has exactly k ergodic acip's μ_1, \dots, μ_k with bounded variation densities,*
- (2) *for every $1 \leq i \leq k$, there exist $k_i \in \mathbb{N}$ and an acip ν_i of f^{k_i} such that*
 - (a)

$$\mu_i = \frac{1}{k_i} \sum_{j=0}^{k_i-1} f_*^j \nu_i$$

- (b) *$(f^{k_i}, f_*^j \nu_i)$ is exact for all j ,*
- (3) *$\text{supp } f_*^j \nu_i$ and $\text{supp } \mu_i$ are both unions of finitely many pairwise disjoint intervals, for all j ,*
- (4) *the union of the basins of μ_1, \dots, μ_k is equal (mod 0) to $[0, 1]$.*

²By [3, Theorem 3.4.3], (f, μ) is exact if and only if for any Borel set $B \subset [0, 1]$ with $\mu(B) > 0$, $\lim_{n \rightarrow \infty} \mu(f^n(B)) = 1$.

Proof. Notice that, by [3, pp. 17 and Theorem 2.3.3], $1/|f'|$ has bounded variation. Thus, Parts (1) and (2) are proved in [3, Theorems 7.2.1 and 8.2.1]. It suffices to prove Part (3) for $\text{supp } f_*^j \nu_i$ which we do applying [3, Theorem 8.2.2] to the acip $f_*^j \nu_i$ of f^{k_i} . Part (4) is proved in [12, Theorem 3.1, Corollary 3.14]. \square

Let

$$\Lambda_{i,j} := \text{supp } f_*^j \nu_i \quad (2.1)$$

We call $\Lambda_{i,1}, \dots, \Lambda_{i,k_i}$ and k_i in Theorem 2.1 the *mixing*³ *components* and the *mixing period* of μ_i , respectively. We also define $\text{Per}(\mu_i) := k_i$.

2.2. Topology. Now we introduce a topology on the space of piecewise expanding maps. Recall that a piecewise expanding map f is defined by a partition $\mathcal{P}_f = \{I_1, \dots, I_m\}$ of the interval $[0, 1]$ with boundary points

$$0 = a_0 < a_1 < \dots < a_{m-1} < a_m = 1. \quad (2.2)$$

To stress the dependence of I_i and a_i on f we shall write $I_i(f)$ and $a_i(f)$, respectively.

Denote by \mathcal{X}_m the set of piecewise expanding maps on m intervals. Given $f, g \in \mathcal{X}_m$ define

$$d(f, g) := \rho(\mathcal{P}_f, \mathcal{P}_g) + \rho_0(f, g) + \rho_{\text{Lip}}(f', g'),$$

where

$$\rho(\mathcal{P}_f, \mathcal{P}_g) := \max_{1 \leq i \leq m-1} |a_i(f) - a_i(g)|$$

$$\rho_0(f, g) := \max_{1 \leq i \leq m} \|f - g \circ \eta_i\|_{C^0(I_i(f))}$$

$$\rho_{\text{Lip}}(f', g') := \max_{1 \leq i \leq m} \{ \|f' - g' \circ \eta_i\|_{\text{Lip}(I_i(f))} + \|g' - f' \circ \eta_i^{-1}\|_{\text{Lip}(I_i(g))} \}$$

and $\eta_i : I_i(f) \rightarrow I_i(g)$ is the affine function that maps $a_i(f)$ to $a_i(g)$. Here, $\|\cdot\|_{C^0}$ and $\|\cdot\|_{\text{Lip}}$ denote the usual norm of a continuous and Lipschitz function, respectively. Clearly, d is a metric, thus (\mathcal{X}_m, d) is a metric space. In fact, (\mathcal{X}_m, d) is a complete metric space.

In this paper, a neighbourhood \mathcal{V} of f is always to be understood in the metric d . Notice that, given any sequence of piecewise expanding maps $f_n \in \mathcal{X}_m$ converging to $f \in \mathcal{X}_m$, f_n also converges to f in the Skorokhod-like metric (cf. [4]).

3. TOPOLOGICAL PROPERTIES

Let f denote a piecewise expanding map. In this section we derive several topological properties of the attractor of f . Some of these properties will be used to prove the combinatorial stability of the attractor (see Theorem 4.1).

³In view of Part (2) of Theorem 2.1, one may be tempted to call these components exact rather than mixing. However, mixing and exactness are equivalent concepts for a piecewise expanding map [3, Theorem 7.2.1].

3.1. **Boundary segments.** Let

$$D = D_f := \{a_1, \dots, a_{m-1}\}, \quad (3.1)$$

where a_i are the points in (2.2) forming the partition \mathcal{P}_f of f . We denote by D_{f^n} the set of points $x \in [0, 1]$ such that $f^k(x) \in D$ for some $0 \leq k < n$. Define $f(x^\pm) := \lim_{y \rightarrow x^\pm} f(y)$ and $f'(x^\pm) := \lim_{y \rightarrow x^\pm} f'(y)$ for every $x \in (0, 1)$. To simplify the presentation, when $x \in \{0, 1\}$ we set $f(x^\pm) = f(x)$ and $f'(x^\pm) = f'(x)$. Similarly, we define $f^n(x^\pm) := \lim_{z \rightarrow x^\pm} f^n(z)$, for $n \geq 0$

Definition 3.1. A *forward orbit* of $x \in [0, 1]$ is a sequence $\{x_n\}_{n \geq 0}$ such that $x_0 = x$ and either $x_n = f^n(x_0^+)$ for every $n \geq 0$ or else $x_n = f^n(x_0^-)$ for every $n \geq 0$. An *orbit segment* starting at $x \in [0, 1]$ and ending at $y \in [0, 1]$ is a finite sequence $\{x_0, \dots, x_n\}$ with $n > 0$ such that $x_0 = x$, $x_n = y$ and either $x_k = f^k(x_0^+)$ for every $k = 0, \dots, n$ or else $x_k = f^k(x_0^-)$ for every $k = 0, \dots, n$. The integer n is called the *length* of the orbit segment.

Notice that any point $x \in (0, 1)$ has exactly two distinct forward orbits if and only if x is a point of discontinuity for some f^n with $n \geq 1$. A point $x \in [0, 1]$ is called *regular* if $x \notin D_{f^n}$ for every $n \geq 1$. Clearly, Lebesgue almost every $x \in [0, 1]$ is regular.

Let μ be an ergodic acip of f and define

$$A_\mu := \text{supp } \mu.$$

Definition 3.2. An orbit segment $\{x_0, \dots, x_n\}$ is called a *boundary segment* of μ if

- (1) $x_0 \in D \cap \text{int}(A_\mu)$,
- (2) $x_i \in \partial A_\mu$, for all $i = 1, \dots, n-1$,
- (3) either $x_n = x_k$ for some $1 \leq k < n$ or else $x_n \in \text{int}(A_\mu)$.

In the following lemma we show that the boundary of A_μ is determined by boundary segments.

Lemma 3.3. *Every $x \in \partial A_\mu$ belongs to a boundary segment of μ .*

Proof. We claim that every $x \in \partial A_\mu$ is contained in an orbit segment $\{x_0, \dots, x_p\}$ starting at a point in $D \cap \text{int}(A_\mu)$ such that $x_k \in \partial A_\mu$ for every $1 \leq k \leq p$. Indeed given $x \in \partial A_\mu$, let

$$E = \{y \in A_\mu : \exists n \in \mathbb{N}, f^n(y^\pm) = x\}.$$

Notice that $\text{int}(A_\mu) \cap E \neq \emptyset$. Indeed, suppose by contradiction that $E \subseteq \partial A_\mu$. Denoting by E_δ a δ -neighbourhood of E in A_μ , since E is finite we have that $f^{-1}(E_\delta) \cap A_\mu \subseteq E_\delta$ for some small enough $\delta > 0$. Thus, $f(A_\mu \setminus E_\delta) \subseteq A_\mu \setminus E_\delta$ which contradicts the ergodicity of μ . So let $y \in \text{int}(A_\mu) \cap E$ such that $f^n(y^\pm) = x$ for the least possible $n \geq 1$. Then $y \in D$, because $f(y^\pm) \in \partial A_\mu$. This proves the claim.

Consider now a forward orbit $\{z_n\}_{n \geq 0}$ of x contained in A_μ . Such a forward orbit always exists since $f(A_\mu \setminus D) \subseteq A_\mu$. If $z_n \in \partial A_\mu$ for all $n \geq 1$, since ∂A_μ is finite, there exists $1 \leq i < n$ such that $z_n = z_i$. Otherwise, there exists $n \geq 1$ such that $z_n \in \text{int}(A_\mu)$. In any case, we obtain a boundary segment containing x . \square

Example 3.4. Consider the tent map $f: [0, 1] \rightarrow [0, 1]$ defined by $f(x) = 2x$ if $x \leq 1/2$ else $f(x) = 2 - 2x$. Notice that f has a unique ergodic acip μ which is the Lebesgue measure on $[0, 1]$. This means that $\partial A_\mu = \{0, 1\}$. The tent map has a single boundary segment $\{1/2, 1, 0\}$.

Lemma 3.5. *Suppose that μ_1 and μ_2 are two distinct ergodic acip's of f . If $A_{\mu_1} \cap A_{\mu_2} \neq \emptyset$, then there exist a boundary segment of μ_1 and a boundary segment of μ_2 both ending at the same point of D or at the same periodic point belonging to $(0, 1)$.*

Proof. Let $x \in \partial A_{\mu_1} \cap \partial A_{\mu_2}$, and suppose that $f^n(x) \notin D$ for all $n \geq 0$, i.e., the forward orbit of x does not contain elements of D . Hence, f^{n+1} is continuous at each $f^n(x)$. Since $f(A_{\mu_i} \setminus D) \subset A_{\mu_i}$ for $i = 1, 2$, it follows that $f^n(x) \in \partial A_{\mu_1} \cap \partial A_{\mu_2} \cap (0, 1)$ for all $n \geq 0$. However, $\partial A_{\mu_1} \cap \partial A_{\mu_2}$ is finite, and so x must be pre-periodic. By Lemma 3.3, the claim follows. \square

3.2. Generic condition. In this section we introduce a generic condition that is sufficient to prove the combinatorial stability in Section 4.

Definition 3.6. We say that f satisfies the *condition* (\star) if there is no orbit segment starting in D and ending at a periodic point in $(0, 1)$ or at a point in D .

Example 3.7. The doubling map $f(x) = 2x \pmod{1}$, $x \in [0, 1]$ satisfies the condition (\star) .

See Figure 1 for an illustration of the condition (\star) . In the following we enumerate several consequences of condition (\star) .

Lemma 3.8. *If f satisfies the condition (\star) , then for every ergodic acips μ, ν of f the following holds:*

- (1) $A_\mu \cap A_\nu = \emptyset$ whenever $\mu \neq \nu$.
- (2) *The mixing components of μ are separated, i.e.,*

$$\Lambda_{i,j} \cap \Lambda_{i',j'} = \emptyset, \quad \text{whenever } (i,j) \neq (i',j').$$
- (3) $D \cap \partial A_\mu = \emptyset$.
- (4) $(0, 1) \cap \partial A_\mu$ does not contain any periodic point.

Proof. Condition (1) follows directly from Lemma 3.5. To prove (2), apply Lemma 3.5 to the ergodic acip's $f_*^j \nu_i$ and $f_*^{j'} \nu_{i'}$ of $f^{k_i k_{i'}}$ where k_i and $k_{i'}$ are the mixing periods of ν_i and $\nu_{i'}$, respectively. By Lemma 3.3,

any point in ∂A_μ belongs to an orbit segment starting in D . So, condition (3) follows from the fact that no orbit segment can start and end in D , and condition (4) follows from the fact that no point in D can be pre-periodic. \square

Next, we show that the condition (\star) is generic in the space of piecewise expanding maps.

Proposition 3.9. *The set of piecewise expanding maps $f \in \mathcal{X}_m$ satisfying the condition (\star) is residual in \mathcal{X}_m .*

Proof. Given integers $p, n \in \mathbb{N}$ and $k \geq 0$ let $\mathcal{Y}_{p,n,k}$ be the set of maps $f \in \mathcal{X}_m$ such that there exists $x \in D_f$ with $f^{n+k}(z) = f^k(z)$ and $z := f^p(x^\pm)$. Similarly, given $p \in \mathbb{N}$ let \mathcal{Z}_p be the set of maps $f \in \mathcal{X}_m$ such that there exist $x \in D_f$ and $z \in D_f$ with $f^p(x^\pm) = z$.

The sets $\mathcal{Y}_{p,n,k}$ and \mathcal{Z}_p are closed with empty interior. Hence, their union \mathcal{W}_m over all integers is a meagre set. Hence, the set of maps $f \in \mathcal{X}_m$ satisfying the condition (\star) is residual. \square

3.3. Saturation. Let $I \subset [0, 1]$ be any open subinterval and consider the open sets $\omega_n(I)$ and $\Omega_n(I)$ defined recursively,

$$\omega_{n+1}(I) = f(\omega_n(I) \setminus D), \quad \omega_0(I) = I,$$

and

$$\Omega_n(I) = \omega_0(I) \cup \cdots \cup \omega_n(I).$$

Also define

$$\Omega(I) := \bigcup_{n=0}^{\infty} \omega_n(I). \quad (3.2)$$

Lemma 3.10. *There exist $\delta > 0$ and $N > 0$ such that for any $n \geq N$, every connected component of $\Omega_n(I)$ has length $\geq \delta$.*

Proof. Let $\delta(n) > 0$ be the minimum length of the connected components of $\Omega_n(I) \setminus D$. Since D is finite and the sequence of open sets $\Omega_n(I)$ is increasing, $D \cap \Omega(I) = D \cap \Omega_{n_0}(I)$ for some $n_0 \geq 0$. Let us show by induction that $\delta(n) \geq \delta(n_0)$ for all $n \geq n_0$. The statement is clearly true for $n = n_0$. Suppose that the inequality is true for a given $n \geq n_0$. Since $\Omega_n(I) \subset \Omega_{n+1}(I)$, for each connected component C of $\Omega_{n+1}(I) \setminus D$, either C contains one connected component of $\Omega_n(I) \setminus D$, or C does not intersect $\Omega_n(I)$, and in this case, it is equal to the image of the union of finitely many connected components of $\Omega_n(I) \setminus D$. In either case, the length of C is greater than or equal to $\delta(n_0)$. The proof is complete. \square

Lemma 3.11. *$\Omega(I)$ is a finite union of intervals.*

Proof. Clearly, $\Omega_n(I)$ is a finite union of intervals. By Lemma 3.10, the set $\Omega_n(I)$ has a lower bound on the size of the connected components for every n sufficiently large. Hence, this implies a similar lower bound

on the size of the connected components of $\Omega(I)$, thus proving the lemma. \square

Lemma 3.12. *If (f, μ) is ergodic, then $A_\mu \setminus \Omega(I)$ is a finite set for every open interval $I \subset A_\mu$.*

Proof. By ergodicity, $\Omega(I) = A_\mu \pmod{0}$. Since $\Omega(I)$ is also a finite union of intervals (see Lemma 3.11) the claim follows. \square

3.4. Periodic orbits. Recall that $[0, 1] = \bigcup_j^m I_j$. Define $\ell(f) = \min_j |I_j|$, where $|I_j|$ denotes the length of I_j .

Lemma 3.13. *Suppose f has least expansion coefficient $\sigma > 2$. Then for every interval $I \subset [0, 1]$, there exist $i \geq 1$ and an open subinterval $W \subset I$ such that*

- (1) $f^i|_W : W \rightarrow \text{int}(I_j)$ is a diffeomorphism, for some $1 \leq j \leq m$,
- (2) $f^{i+1}(W)$ is an open interval and $|f^{i+1}(W)| \geq \sigma \ell(f)$.

Proof. Part (1). Let $B = \bigcup_{j=1}^m \partial I_j$. We claim that given any interval $I \subset [0, 1]$, there exist $i \in \mathbb{N}$ and a subinterval $W = (a, b) \subset I$ with $a, b \in f^{-i}(B)$ such that

$$W \cap f^{-k}(B) = \emptyset \quad \text{for } 0 \leq k \leq i.$$

Indeed, if this was not the case, then for every $i \geq 1$, no two consecutive points of $I \cap (B \cup f^{-1}(B) \cup \dots \cup f^{-i}(B))$ would belong to $f^{-i}(B)$. It is not difficult to see that this would imply that $f^i(I)$ consists of at most 2^i intervals. But $\sigma > 2$, and so the length of one of these intervals would be not less than $(\sigma/2)^i \rightarrow +\infty$, as $i \rightarrow +\infty$, giving a contradiction. By the definition of B , we have $f^i(W) = \text{int } I_j$ for some j .

Part (2). From Part (1), it follows that $f^{i+1}(W) = f(\text{int } I_j)$ is an open interval, and so

$$|f^{i+1}(W)| = |f(I_j)| \geq \sigma |I_j| \geq \sigma \ell(f).$$

\square

Let μ be an ergodic acip of f . In the next theorem we prove, using elementary methods, that the periodic points of f are dense in the support of μ .

Theorem 3.14. *The periodic points of f are dense in A_μ .*

Proof. To obtain the wanted conclusion, we show that every open interval $U \subset A_\mu$ contains a periodic point of f . Let $k \in \mathbb{N}$ such that f^k has least expansion coefficient > 2 . Also, let \mathcal{I}_μ be the collection of the connected components of $\text{int}(A_\mu \setminus D_{f^k})$.

By Lemma 3.12, we can assume that $U \subset \Omega_N(I)$ for some large enough integer N and all $I \in \mathcal{I}_\mu$. Then reducing the open interval U even further we can assume that for every $I \in \mathcal{I}_\mu$ there exists a positive integer $n_I \leq N$ such that $U \subset \omega_{n_I}(I)$. We conclude that $f^{n_I}|_{I'} : I' \rightarrow U$ is a diffeomorphism for some open subinterval $I' \subset I$.

By Lemma 3.13, there exists an open subinterval $W \subset U$ and $i \geq 1$ such that $g := f^{ik}|_W : W \rightarrow I$ is a diffeomorphism for some $I \in \mathcal{I}_\mu$.

Let $W' := g^{-1}(I)$. Then $f^{ik+n_I}|_{W'} : W' \rightarrow U$ is a diffeomorphism and $W' \subset W \subset U$. Since f^{ik+n_I} is expanding, it has a fixed point inside U . This proves the theorem. \square

When f is continuous, the density of periodic points can be deduced from the work of Hofbauer [7] on continuous piecewise monotonic maps.

In the next result we give a characterization for (f, μ) to be exact in terms of periodic orbits.

Proposition 3.15. *(f, μ) is exact if and only if any open set in A_μ contains two periodic points having coprime periods.*

Proof. If (f, μ) is not exact, then it has $k \geq 2$ mixing components. Therefore, there is an open set U in A_μ where k must divide the period of any periodic point in U . This shows that the coprimality condition of the periods is sufficient for exactness. To show that it is necessary, suppose that (f, μ) is exact. By exactness, $\mu(f^n(I)) \rightarrow 1$ as $n \rightarrow \infty$ for any interval $I \subset [0, 1]$. Let U be an open subinterval of A_μ . Shrinking U if necessary, for every $I \in \mathcal{I}_\mu$ there exists a positive integer $n_I \geq 1$ such that $U \subset \omega_{n_I}(I)$ and $U \subset \omega_{n_I+1}(I)$. Here, \mathcal{I}_μ denotes the collection of the connected components of $\text{int}(A_\mu \setminus D)$. Arguing as in the proof of Theorem 3.14, we conclude that both f^{i+n_I} and f^{i+n_I+1} have a fixed point inside U for some $I \in \mathcal{I}_\mu$ and integer $i \geq 1$. Since U is arbitrary, this shows the existence of two periodic orbits with coprime periods in any open set in A_μ . \square

Remark 3.16. The existence of a fixed point in A_μ is not sufficient to show that (f, μ) is exact. Indeed, consider the orientation-reversing Lorenz map $f(x) = 1 - a(x - 1/2) \pmod{1}$ with $a = \sqrt{2}$.

Remark 3.17. Let μ be an ergodic acip with separated mixing components (see Part (2) of Lemma 3.8). If A_μ contains two periodic points with coprime periods, then (f, μ) is exact.

4. COMBINATORIAL STABILITY

In this section, we prove that a piecewise expanding map f is combinatorially stable provided it satisfies the condition (\star) . Recall by Proposition 3.9, the condition (\star) is generic in the space of piecewise expanding maps on m intervals.

Let $\mathcal{E}(f)$ denote the finite set of ergodic acip's of f .

Theorem 4.1. *If $f \in \mathcal{X}_m$ satisfies the condition (\star) , then there is a neighbourhood \mathcal{V} of f in \mathcal{X}_m such that for every $g \in \mathcal{V}$, there is a bijection Θ_g between $\mathcal{E}(f)$ and $\mathcal{E}(g)$ satisfying $\Theta_f = \text{id}$ and $\text{Per}(\Theta_g(\mu)) = \text{Per}(\mu)$ for every $\mu \in \mathcal{E}(f)$ and $g \in \mathcal{V}$. Furthermore, for every $\mu \in \mathcal{E}(f)$, the map $\mathcal{V} \ni g \mapsto A_{\Theta_g(\mu)}$ is continuous at f with respect to the Hausdorff metric.*

Under the generic condition (\star) , Theorem 4.1 shows that the number of ergodic acip's of f and the corresponding mixing periods do not change in a neighbourhood of f , a property we call *combinatorial stability*.

Theorem 4.1 is proved in Section 4.2. In the following we prove several preliminary lemmas.

4.1. Perturbation lemmas. The following lemma follows from standard considerations in hyperbolic theory. For the convenience of the reader we include here a proof.

Lemma 4.2. *Let $x \in [0, 1]$ be a regular periodic point of f . There is a neighbourhood \mathcal{V} of f and a continuous map $g \mapsto x_g$ defined on \mathcal{V} such that x_g is a periodic point of g having the same period of x .*

Proof. Let $k > 0$ be the period of x . Since f is expanding there is an interval J containing x and with closure not intersecting D such that $f^k(J)$ is an interval, $J \subset f^k(J)$ and $h_f := f^k|_J$ is a bijection. By continuity, there is a neighbourhood \mathcal{V} of f such that the same holds for every $g \in \mathcal{V}$, in particular $h_g := g^k|_J$ is a bijection. Consider the map $\varphi: \mathcal{V} \times J \rightarrow J$ defined by $\varphi(g, x) = h_g^{-1}(x)$. Clearly, φ is continuous and $\varphi(g, \cdot)$ is a uniform contraction. Therefore, by the contraction fixed point theorem (continuous dependence of parameters version), $\varphi(g, \cdot)$ has a unique fixed point x_g which depends continuously on $g \in \mathcal{V}$. \square

We say that x_g is the continuation of x by g .

Let μ be an ergodic acip of f . In the following lemma we use the boundary segments associated to ∂A_μ to define a closed forward invariant set for small perturbations of f .

Lemma 4.3. *If $D \cap \partial A_\mu = \emptyset$ and $(0, 1) \cap \partial A_\mu$ does not contain periodic points, then there exist a neighbourhood \mathcal{V} of f and a continuous⁴ map $g \mapsto U_\mu(g)$ defined on \mathcal{V} such that*

- (1) $U_\mu(f) = A_\mu$,
- (2) $U_\mu(g)$ is a finite union of closed intervals for every $g \in \mathcal{V}$,
- (3) $g(U_\mu(g)) \subseteq U_\mu(g)$ for every $g \in \mathcal{V}$.

Proof. Notice that, for every $x \in D \cap \text{int}(A_\mu)$ we have $f(x^\pm) \in A_\mu$. By slightly abusing notation we shall call the points in $D \cap \text{int}(A_\mu)$ together with their images that satisfy $f(x^\pm) \in \text{int}(A_\mu)$ also boundary segments. Consider the collection \mathcal{B} of all boundary segments of μ . Clearly, given $d \in D \cap \text{int}(A_\mu)$ there is $\gamma = \{x_0, \dots, x_n\} \in \mathcal{B}$ such that $x_0 = d$. Moreover, by Lemma 3.3, every point in ∂A_μ is contained in a boundary segment in \mathcal{B} . Notice that, two or more points in ∂A_μ may be covered by a single boundary segment and a single point in ∂A_μ may be covered by more than one boundary segment.

⁴In the Hausdorff metric.

By the hypothesis $D \cap \partial A_\mu = \emptyset$, we have $f(A_\mu) \subseteq A_\mu$. This together with the hypothesis $(0, 1) \cap \partial A_\mu$ has no periodic points, implies that given any boundary segment $\gamma = \{x_0, \dots, x_n\} \in \mathcal{B}$ it satisfies the properties:

- (1) $x_1 = f(x_0^\pm)$ and $x_0 \in D \cap \text{int}(A_\mu)$,
- (2) $x_{i+1} = f(x_i)$ for each $i = 1, \dots, n-1$,
- (3) $x_i \in \partial A_\mu$ and $x_i \notin D$ for each $i = 1, \dots, n-1$,
- (4) one of the following alternatives hold:
 - (a) $x_n \in \text{int}(A_\mu)$,
 - (b) $x_n = x_{n-2} \in \{0, 1\}$,
 - (c) $x_n = x_{n-1} \in \{0, 1\}$.

For each $x \in \partial A_\mu$ let $\mathcal{B}_x \subset \mathcal{B}$ denote the collection of all boundary segments passing through x . By previous observations, $\mathcal{B}_x \neq \emptyset$. So we may define the *order* of $x \in \partial A_\mu$ to be

$$\text{ord}(x) := \max\{k \in \mathbb{N} : x_k = x \text{ for some } \{x_0, \dots, x_n\} \in \mathcal{B}_x\}.$$

It is convenient to set $\text{ord}(x) = 0$ for any $x \in D \cap \text{int}(A_\mu)$. Notice that, $\text{ord}(z) \leq \text{ord}(x)$ whenever $x = f(z^\pm)$ with $x, z \in \partial A_\mu \cup (D \cap \text{int}(A_\mu))$.

Define $E := \partial A_\mu \cup (D \cap \text{int}(A_\mu))$. The points in E induce a partition of A_μ as a union of closed intervals,

$$A_\mu = [\alpha_0, \beta_0] \cup [\alpha_1, \beta_1] \cdots \cup [\alpha_q, \beta_q],$$

where $\alpha_0 < \beta_0 \leq \alpha_1 < \beta_1 \leq \cdots \leq \alpha_q < \beta_q$ and $\alpha_i, \beta_i \in E$. Notice that two consecutive intervals are either disjoint or intersect at a single point belonging to $D \cap \text{int}(A_\mu)$.

Now, let $g \in \mathcal{V}$ where \mathcal{V} is an ε -neighbourhood of f and $\varepsilon > 0$ is sufficiently small. We define a map $\varphi_g : E \rightarrow [0, 1]$ in the following way. Given $x \in D \cap \text{int}(A_\mu)$, we set $\varphi_g(x) := a_i(g)$ where $x = a_i(f)$ for some $1 \leq i \leq m-1$. Otherwise, suppose that $x \in \partial A_\mu$. We have two cases. Either x is periodic under f or x is not periodic. In the first case, $x \in \{0, 1\}$, and we set $\varphi_g(x) = x$. In the second case, we will define φ_g on E inductively on the order of the points. So suppose that φ_g has been defined for points in E whose order is $\leq n$. Let $x \in E$ such that $\text{ord}(x) = n+1$. We suppose that x is a left boundary point of ∂A_μ , i.e., $[x, x+\delta] \subset A_\mu$ for $\delta > 0$ small. The case of x being a right boundary of A_μ is treated similarly (the min below becomes a max). Then we define

$$\varphi_g(x) := \min\{g(\varphi_g(z)^\pm) : z \in E, x = f(z^\pm)\}.$$

In this way we have a well-defined map $\varphi_g : E \rightarrow [0, 1]$. Choosing $\varepsilon > 0$ smaller, if necessary, φ_g becomes injective. Moreover, $\varphi_f = \text{id}$. Using the map φ_g we finally define,

$$U_\mu(g) := [\varphi_g(\alpha_0), \varphi_g(\beta_0)] \cup \cdots \cup [\varphi_g(\alpha_q), \varphi_g(\beta_q)].$$

Now it is simple to check that $U_\mu(g)$ satisfies all properties stated in the lemma. \square

Recall that $[0, 1] = \bigcup_j^m I_j$ and $\ell(f) = \min_j |I_j|$, where $|I_j|$ denotes the length of I_j .

Lemma 4.4. *If f has no orbit segment connecting points in D , then there is a neighbourhood \mathcal{V} of f and a constant $\eta = \eta(\mathcal{V}) > 0$ such that for every interval $I \subset [0, 1]$ there exists $n \geq 1$ for which $g^n(I)$ contains an open interval of length greater than 2η for every $g \in \mathcal{V}$.*

Proof. Apply Lemma 3.13 to f^k with $k > 0$ being the smallest integer such that the least expansion of f^k is greater than 2. Then $n = ik$, where i is as in Lemma 3.13 (applied to f^k), and $\eta(f) = \ell(f^k)$.

Let $\eta := \min_{g \in \mathcal{V}} \eta(g)$. Because f has no orbits segments connecting points in D , for every g sufficiently close to f , the set D_{g^k} has the same cardinality and is close to the set D_{f^k} , i.e., no discontinuity of f^k bifurcates into 2 or more distinct discontinuities of g^k . Thus, $\eta > 0$ and the integer n can be made uniform (not depending on g). The conclusion of the lemma follows. \square

Remark 4.5. In the previous lemma it is sufficient to assume the weaker hypothesis that f has no segments of order less or equal than k connecting points in D .

Definition 4.6. Given points $x, y \in [0, 1]$ we say that x leads to y under f , and write $x \rightsquigarrow y$, if for every neighbourhood V of x there exists $n \geq 0$ such that $y \in f^n(V)$. We say that two points are heteroclinically related under f if $x \rightsquigarrow y$ and $y \rightsquigarrow x$.

Clearly, the heteroclinic relation is an equivalence relation. Another key observation is that the heteroclinic relation between periodic points is stable under perturbation.

Lemma 4.7. *If two regular periodic points x and y of f are heteroclinically related under f , then there is a neighbourhood \mathcal{V} of f such that for every $g \in \mathcal{V}$ the continuations x_g and y_g of the periodic points x and y are also heteroclinically related under g .*

Proof. Let x and y be two regular periodic points for f such that $x \rightsquigarrow y$. By Lemma 4.2, x and y have continuations x_g and y_g for every $g \in \mathcal{V}$ where \mathcal{V} is a neighbourhood of f . We will show that $x_g \rightsquigarrow y_g$. Denote by p the period of x and x_g . Define

$$\tau := \frac{1}{2} \inf_{g \in \mathcal{V}} \text{dist}(x_g, D_{g^p}) > 0.$$

Notice that $I_\tau(x_g) \cap D_{g^p} = \emptyset$ for every $g \in \mathcal{V}$ where $I_\tau(z) := (z - \tau, z + \tau)$. Since $x \rightsquigarrow y$, there is $n = n(x, \tau) \geq 0$ such that $y \in f^n(I_\tau(x))$. Shrinking \mathcal{V} if necessary, we may assume that $y_g \in g^n(I_\tau(x_g))$ for every $g \in \mathcal{V}$. Now let J be any interval containing x_g . Since g is expanding, there is $k \geq 0$ such that $I_\tau(x_g) \subset g^{kp}(J)$. Thus, $y_g \in g^{kp+n}(J)$. This shows that $x_g \rightsquigarrow y_g$. \square

4.2. Proof of Theorem 4.1. Let $\mu \in \mathcal{E}(f)$ be an ergodic acip of a piecewise expanding map f satisfying the condition (\star) .

We divide the proof of Theorem 4.1 in four lemmas. Throughout the proof, we assume that the neighbourhood \mathcal{V} of f is chosen to be sufficiently small so that the hypothesis of the perturbation lemmas are verified.

Lemma 4.8. *There is a neighbourhood \mathcal{V} of f such that for each $g \in \mathcal{V}$, there exists a unique $\nu \in \mathcal{E}(g)$ such that $A_\nu \subseteq U_\mu(g)$.*

Proof. By Lemma 3.8, f satisfies the hypothesis of Lemma 4.3 and Lemma 4.4. Let \mathcal{V} be a neighbourhood of f for which the conclusion of Lemma 4.3 holds. Given $g \in \mathcal{V}$, the existence of $\nu \in \mathcal{E}(g)$ such that $A_\nu \subseteq U_\mu(g)$ follows directly from Part (3) of Lemma 4.3 and Theorem 2.1.

To prove the uniqueness, suppose that ν_1 and ν_2 are two ergodic acips of $g \in \mathcal{V}$ whose supports are contained in $U_\mu(g)$. We want to show that $\nu_1 = \nu_2$.

Take $\eta = \eta(\mathcal{V}) > 0$ given by Lemma 4.4. Let $k \in \mathbb{N}$ such that f^k has least expansion coefficient > 2 . Also let \mathcal{I}_μ be the set of connected components of $\text{int}(A_\mu \setminus D_{f^k})$ and Ω the intersection of $\Omega(I)$ over all I belonging to \mathcal{I}_μ . Recall that $\Omega(I)$ is defined in (3.2). By Lemma 3.12, Ω equals A_μ except for a finite set of points, which we denote by E . Since periodic points are dense in A_μ (by Theorem 3.14), we can take a $\eta/3$ -dense set $X := \{x_1, \dots, x_r\} \subset \Omega \setminus D$ of regular periodic points of f such that x_i and x_j are heteroclinically related under f for all $1 \leq i, j \leq r$. Indeed, by Lemma 3.13, for any $x \in X$ there is $I \in \mathcal{I}_\mu$ such that $I \subset f^{nk}(V)$ for some $n \in \mathbb{N}$ and neighbourhood V of x . But $\Omega(I)$ contains $A_\mu \setminus E$. Therefore, $x \rightsquigarrow y$ for any $y \in X$.

Let $X' := \{x'_1, \dots, x'_r\}$ denote the set of continuations of the periodic points in X for some nearby map $g \in \mathcal{V}$. According to Lemma 4.7, by choosing \mathcal{V} sufficiently small, we can assume that x'_i and x'_j are heteroclinically related under g for all $1 \leq i, j \leq r$. We can also assume that $|x_j - x'_j| < \eta/3$ for all $j = 1, \dots, r$ and that the Hausdorff distance between $U_\mu(f)$ and $U_\mu(g)$ is also less than $\eta/3$. This follows from the continuity of the maps in Lemma 4.2 and Lemma 4.3.

Take now points $y_i \in A_{\nu_i} \subseteq U_\mu(g)$ and neighbourhoods V_i of y_i in A_{ν_i} . By Lemma 4.4, there exist subintervals $I_i \subset V_i$, integers $n_i \geq 1$ and points $z_i \in U_\mu(g)$ such that $g^{n_i}(I_i) = (z_i - \eta, z_i + \eta)$. Because of the previous considerations,

$$\begin{aligned} \text{dist}(z_i, X') &\leq \text{dist}(U_\mu(g), U_\mu(f)) + \text{dist}(U_\mu(f), X) + \text{dist}(X, X') \\ &< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta. \end{aligned}$$

Thus, $g^{n_i}(I_i) \cap X' \neq \emptyset$ which implies that y_i leads to a periodic point in X' under g . Finally, since the points $y_i \in A_{\nu_i}$ are arbitrary, and

all points in X' are heteroclinically related under g , it follows that $\nu_1 = \nu_2$. \square

Lemma 4.8 shows that we have a well-defined map for every $g \in \mathcal{V}$,

$$\Theta_g: \mathcal{E}(f) \rightarrow \mathcal{E}(g), \quad \mu \mapsto \nu,$$

where $\nu \in \mathcal{E}(g)$ comes from Lemma 4.8. Clearly, $\Theta_f = \text{id}$. In the following two lemmas we show that Θ_g is a bijection as stated in Theorem 4.1.

Lemma 4.9. *The map Θ_g is one-to-one.*

Proof. By Part 1 of Lemma 3.8, $A_{\mu_1} \cap A_{\mu_2} = \emptyset$ for every $\mu_1 \neq \mu_2$ in $\mathcal{E}(f)$. We can further suppose that \mathcal{V} is sufficiently small so that $U_{\mu_1}(g) \cap U_{\mu_2}(g) = \emptyset$ for every $g \in \mathcal{V}$ and every $\mu_1 \neq \mu_2$ in $\mathcal{E}(f)$. Now suppose that $\nu = \Theta_g(\mu_1) = \Theta_g(\mu_2)$. But, $A_\nu \subseteq U_{\mu_i}(g)$ which can only happen if $\mu_1 = \mu_2$. \square

Lemma 4.10. *The map Θ_g is onto.*

Proof. By Part (3) of Theorem 2.1, the union B of all basins of attraction $B(\mu_i)$ over the elements $\mu_i \in \mathcal{E}(f)$ coincides with the interval $[0, 1]$ up to a zero Lebesgue measure set. By Lemma 3.8, f satisfies the hypothesis of Lemma 4.4. Let $\eta = \eta(\mathcal{V}) > 0$ be the constant in Lemma 4.4.

Given $\mu \in \mathcal{E}(f)$, let φ be a continuous function having compact support inside $\text{int}(A_\mu)$. It follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu > 0, \quad \forall x \in B(\mu).$$

Hence, for every $x \in B(\mu)$ there are infinitely many integers $k_i \geq 0$ such that $f^{k_i}(x) \in \text{int}(A_\mu)$. This implies that there is an $\eta/2$ -dense set $Z := \{z_1, \dots, z_r\} \subset B$ of $[0, 1]$ such that for every $z_i \in Z$, we can find $k \in \mathbb{N}$ and $\mu \in \mathcal{E}(f)$ for which $f^k(z_i) \in \text{int}(A_\mu)$.

Now, let $\mu' \in \mathcal{E}(g)$ and take $y \in A_{\mu'}$. Also let W be a neighbourhood of y in $A_{\mu'}$. By Lemma 4.4, there is $n \in \mathbb{N}$ such that $g^n(W)$ contains an interval of length greater than or equal to η . Thus, $g^n(W)$ contains in its interior a point $z_i \in Z$, i.e., y leads to z_i under g . It follows that the intersection $g^{n+t_i}(W) \cap U_{\mu_{j_i}}(g)$ contains an interval for some $t_i \geq 0$ and $1 \leq j_i \leq \#\mathcal{E}(f)$. Arguing as in the proof of Lemma 4.8, this shows that $\mu' = \Theta_g(\mu_{j_i})$, and thus Θ_g is onto. \square

It remains to show that f and $g \in \mathcal{V}$ have the same number of mixing components.

Lemma 4.11. *$\text{Per}(\Theta_g(\mu)) = \text{Per}(\mu)$ for every $\mu \in \mathcal{E}(f)$ and $g \in \mathcal{V}$.*

Proof. Consider an ergodic acip μ for f . Let $k := \text{Per}(\mu)$ and denote by Λ_i with $1 \leq i \leq k$, the mixing components of μ . We can write,

$$U_\mu(g) = U_\mu^{(1)}(g) \cup \dots \cup U_\mu^{(k)}(g)$$

where $U_\mu^{(i)}(g)$ are the continuations of $\Lambda_i \subset U_\mu(f) = A_\mu$, i.e., the connected components of $U_\mu(g)$ which intersect Λ_i and $U_\mu^{(i)}(f) = \Lambda_i$. Since f satisfies Part 2 of Lemma 3.8, the sets $U_\mu^{(i)}(g)$ are pairwise disjoint for every $g \in \mathcal{V}$.

Let $\mu' := \Theta_g(\mu)$ be the unique ergodic acip for $g \in \mathcal{V}$ such that $A_{\mu'} \subseteq U_\mu(g)$ and define $k' := \text{Per}(\mu')$. We first notice that k divides k' because

$$g^k(U_\mu^{(j)}(g)) \subseteq U_\mu^{(j)}(g), \quad \forall j = 0, \dots, k-1.$$

To prove that $k = k'$ we will assume without loss of generality that $k = 1$. For the general case we can replace g by g^k , resp. f by f^k . So we suppose that f has a unique mixing component A_μ , i.e., (f, μ) is exact.

Let $\eta = \eta(\mathcal{V}) > 0$ be the constant in Lemma 4.4 and $X \subset A_\mu$ be a finite set of regular periodic points of f with the property that every sub-interval $J \subset A_\mu$ of length greater or equal than $\eta/2$ contains at least two periodic points in X with coprime periods. This is possible by Proposition 3.15.

Shrinking the neighbourhood \mathcal{V} if necessary, we may assume that $A_\mu \cap A_{\mu'}$ contains an interval I whose length is $\geq \eta/2$. Thus, I contains two periodic points x and y in X with coprime periods. According to Lemma 4.2, these periodic points have continuations $x_g, y_g \in I$ for every $g \in \mathcal{V}$ whose periods are coprime as well. Thus, by Remark 3.17, we conclude that (g, μ') is exact. □

Finally, to complete the proof of Theorem 4.1, it remains to prove that the map $g \ni \mathcal{V} \mapsto A_{\Theta_g(\mu)}$ is continuous at f . By Lemma 4.3 and because $A_\mu = U_\mu(f)$, the map $g \ni \mathcal{V} \mapsto A_{\Theta_g(\mu)}$ is upper semi-continuous at f . The lower semi-continuity at f follows from the density of periodic points (Theorem 3.14) and the fact that any finite set of heteroclinically related regular periodic points is stable (Lemma 4.7).

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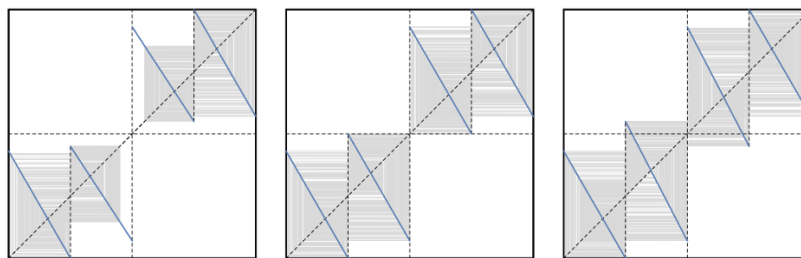
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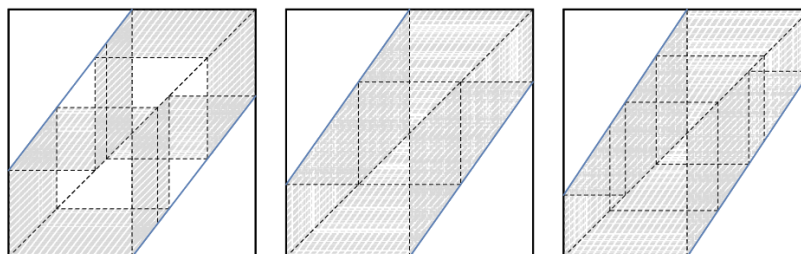
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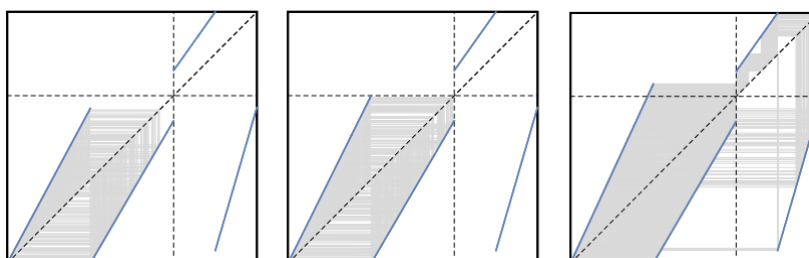
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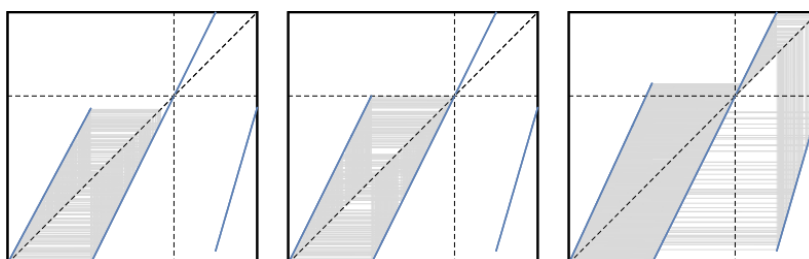
(A) A family of piecewise expanding maps where two ergodic acips collide. The middle map has an orbit segment connecting two discontinuous points.



(B) The discontinuity of the Lorenz map $f_a(x) = a(x - 1/2) \pmod{1}$ with $a = \sqrt{2}$ is pre-periodic. For $a < \sqrt{2}$ the Lorenz family has one ergodic acip of period 2, which becomes exact for $a > \sqrt{2}$.



(C) A family of piecewise expanding maps where the support of the acip explodes. The middle map has a discontinuity at a boundary point of the acip's support.



(D) A family of piecewise expanding maps where the support of the acip explodes. The middle map has a fixed point $x > 1/2$ which is a boundary point of the acip's support.

FIGURE 1. Examples of families of piecewise expanding maps where the middle map does not satisfy the condition (\star) .