Renormalization of Diophantine Skew Flows, with Applications to the Reducibility Problem

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Abstract. We introduce a renormalization group framework for the study of quasiperiodic skew flows on Lie groups of real or complex $n\times n$ matrices, for arbitrary Diophantine frequency vectors in $\mathbb{R}^d$ and dimensions d,n. In cases where the group component of the vector field is small, it is shown that there exists an analytic manifold of reducible skew systems, for each Diophantine frequency vector. More general near-linear flows are mapped to this case by increasing the dimension of the torus. This strategy is applied for the group of unimodular 2×2 matrices, where the stable manifold is identified with the set of skew systems having a fixed fibered rotation number. Our results apply to vector fields of class C^γ, with γ depending on the number of independent frequencies, and on the Diophantine exponent.

1. Introduction and main results

Let $\mathfrak{G}$ be a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{R})$, and denote by $\mathfrak{A}$ the corresponding Lie algebra. We consider vector fields on $\Lambda = \mathbb{T}^d \times \mathfrak{G}$ of the form

$$X(q,y) = \bigl(\omega, f(q)y\bigr)\,, \qquad f(q)\in\mathfrak{A}\,, \qquad (q,y)\in\Lambda\,. \tag{1.1}$$

Here, $\mathbb{T}^d$ denotes the d-torus, with $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$. Such a vector field X determines a linear flow on the torus, $q(t)=q_0+t\omega$, and a linear evolution equation on $\mathfrak{G}$,

$$\dot y(t) = f(q_0+t\omega)y(t)\,, \qquad y(0)=y_0\,, \tag{1.2}$$

whose coefficients are periodic or quasiperiodic functions of t, depending on the frequency vector ω. If $t\mapsto \Phi_X^t(q_0)$ denotes the solution of (1.2), for the case where $y_0\in\mathfrak{G}$ is the identity, then the flow Ψ_X associated with the vector field (1.1) can be written as

$$\Psi_X^t(q_0,y_0) = \bigl(q_0+t\omega, \Phi_X^t(q_0)y_0\bigr)\,, \qquad (q_0,y_0)\in\Lambda\,, \quad t\in\mathbb{R}\,. \tag{1.3}$$

Such flows are commonly referred to as *skew flows.* Classical Floquet theory shows that if $t\mapsto q(t)$ is periodic, and in particular if $d=1$, then the system is reducible. To be more precise, the vector field (1.1) is said to be *reducible* if there exists a function $V:\mathbb{T}^d\to\mathfrak{G}$, such that

$$\Phi_X^t(q) = V(q+t\omega)^{-1}e^{tC}V(q)\,, \qquad t\in\mathbb{R}\,, \quad q\in\mathbb{T}^d\,, \tag{1.4}$$

for some constant matrix $C\in\mathfrak{A}$. If $\omega\in\mathbb{R}^d$ is fixed, we will also refer to f as being reducible. Another characterization of reducibility can be given by considering the map $\mathcal{V}:\Lambda\to\Lambda$, defined by

$$\mathcal{V}(q,y) = \bigl(q, V(q)y\bigr)\,. \tag{1.5}$$

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The pushforward of $X = (\omega, f\,.)$ under this map is given by the equation

$$\big(\mathcal{V}_* X\big)(q,y) = \big(\omega, (\mathcal{V}_\star f)(q)y\big)\,, \qquad \mathcal{V}_\star f = (D_\omega V + Vf)V^{-1}\,, \tag{1.6}$$

where $D_\omega = \omega\cdot\nabla$. Modulo smoothness assumptions, (1.4) is equivalent to $\mathcal{V}_\star f \equiv C$.

More recent results concern the reducibility of skew systems with rationally independent frequencies $\omega_1,\ldots,\omega_d$, where $t\mapsto q(t)$ is quasiperiodic. For such systems, solving $\mathcal{V}_\star f \equiv C$ leads to small divisor problems, as in classical KAM theory. Results based on KAM type methods have been obtained in the case where $\mathfrak{G} = \mathrm{SL}(2,\mathbb{R})$ [1,2,3], motivated by the study of the one-dimensional Schrödinger equation with quasiperiodic potential, and for compact Lie groups [4,5]. In particular, Eliasson's result [3] for $\mathfrak{G} = \mathrm{SL}(2,\mathbb{R})$ guarantees reducibility for analytic vector fields of the form (1.1), with ω Diophantine, and with the fibered rotation number (associated with a rotation in $\mathfrak{G}$) being either rational or Diophantine with respect to ω. The vector field is required to be close to constant, but the smallness condition does not depend on further arithmetic properties of the rotation number. By contrast to these results, there are also generic examples of non-reducible systems [8,3,9].

Another approach to the reducibility problem involves renormalization methods. For discrete time cocycles over rotations by an irrational angle α, and for $\mathfrak{G} = \mathrm{SU}(2)$, Rychlik introduced in [8] a renormalization scheme based on a rescaling of first return maps, using the continued fractions expansion of α. Later, Krikorian improved the method in [6,7], where he was able to prove global (non-perturbative) results for compact C^∞ cocycles. A non-compact case was treated in [10]. In the context of flows, renormalization techniques were used in [11] to prove a local normal form theorem for analytic skew systems with a Brjuno base flow. Unlike the KAM methods, the renormalization approach has so far been restricted to skew systems with a one-dimensional base map or two-dimensional base flow.

In this paper, we introduce a new renormalization group approach, which allows us to extend the analysis of near-constant skew flows in several directions. One of its characteristics is that fibered rotation numbers are included in the renormalization procedure. This leads naturally to multi-frequency problems, and to the analysis of skew systems over tori of arbitrary dimensions, which we handle by making use of the multidimensional continued fractions algorithm introduced in [19]. In addition, we reduce the smoothness condition on the vector field, by requiring only a finite degree of differentiability.

We focus on cases where $\omega\in\mathbb{R}^d$ is *Diophantine*, in the sense that

$$|\omega\cdot\nu| \ge C\|\nu\|^{1-d-\beta}\,, \qquad \nu\in\mathbb{Z}\setminus\{0\}\,, \tag{1.7}$$

for some constants $\beta, C > 0$. It is well known that for any fixed $\beta > 0$, the measure of the set of vectors ω that violate (1.7) approaches zero as C tends to zero [20]. The constants $\beta, C > 0$ are considered fixed in the rest of this paper.

Our vector fields are assumed to be of class C^γ, with γ larger than some constant $\gamma_0(\beta)$ specified below. Given any $\gamma\ge 0$, define $\mathcal{F}_\gamma$ to be the Banach space of integrable functions $f:\mathbb{T}^d\to\mathrm{GL}(n,\mathbb{C})$, for which the norm

$$\|f\|_\gamma = \|f_0\| + \sum_{0\ne\nu\in\mathbb{Z}^d}\|f_\nu\|(2\|\nu\|)^\gamma\,, \qquad f_\nu = (2\pi)^{-d}\int_{\mathbb{T}^d} f(q)e^{-i\nu\cdot q}dq\,, \tag{1.8}$$

is finite. Here, and in what follows, we use the standard ℓ^2 norm on the spaces $\mathbb{C}^m$, and the corresponding operator norm for $m \times m$ matrices. Define $\mathbb{E}f$ to be the torus-average f_0 of a function $f \in \mathcal{F}_\gamma$. The set of functions in $\mathcal{F}_\gamma$ that take values in $\mathfrak{G}$ or $\mathfrak{A}$ will be denoted by $\mathcal{G}_\gamma$ or $\mathcal{A}_\gamma$, respectively.

Our first result describes a class of vector fields $X = (\omega, f\,.)$ that are reducible to the trivial vector field $(\omega, 0)$. Define

$$\gamma_0(\beta) = (d+\beta)\left[1 + 2\beta + 2\sqrt{\beta\bigl[1+\beta-1/(d+\beta)\bigr]}\,\right] - 1\,. \tag{1.9}$$

Theorem 1.1. *Given $\gamma \geq \gamma_2 > \gamma_0(\beta)$, there exists an open neighborhood B of the origin in $\mathcal{F}_\gamma$, and for each Diophantine unit vector ω satisfying (1.7) a manifold $\mathcal{M}$ in B, such that the following holds. $\mathcal{M}$ is the graph of an analytic map $M : (\mathbb{I}-\mathbb{E})B \to \mathbb{E}B$, which vanishes together with its derivative at the origin, and which takes values in $\mathcal{A}_\gamma$ when restricted to $\mathcal{A}_\gamma$. Every function f on $\mathcal{M}$ is reducible to zero. The corresponding change of coordinates V belongs to $\mathcal{F}_\varepsilon$ and depends analytically on f, where $\varepsilon = \gamma - \gamma_2$. If in addition, $f \in \mathcal{A}_\gamma$, then V belongs to $\mathcal{G}_\varepsilon$, and if f is the restriction to $\mathbb{T}^d$ of an analytic function, then so is V.*

Here, a function ψ defined on $\mathcal{M}$ is said to be analytic if $\psi \circ M$ is analytic on the domain of M.

This theorem can also be applied to vector fields $Y = (w, g\,.)$, whose group component g is close to a constant matrix A, but not necessarily small. But w and A have to satisfy a certain Diophantine condition. More specifically, assume that $A \in \mathfrak{A}$ admits a spectral decomposition $A = \kappa \cdot J = \kappa_1 J_1 + \ldots + \kappa_\ell J_\ell$, where κ is some vector in $\mathbb{R}^\ell$, and where the J_j are linearly independent mutually commuting matrices in $\mathfrak{A}$, such that $t \mapsto \exp(tJ_j)$ is 2π-periodic. The vector κ will be referred to as the frequency vector of A.

In order to see how Theorem 1.1 can be applied to $g \approx A$, we start with a skew system $Y = (w, g\,.)$ on $\mathbb{T}^m \times \mathfrak{G}$, and then take $d = m + \ell$. Clearly, if $g \equiv A = \kappa \cdot J$, then the flow for Y is equivalent to the flow for $X = (\omega, 0)$, with $\omega = (w, \kappa)$. More generally, if $g - A$ is small but not necessarily zero, we consider the function

$$f(q) = e^{-r\cdot J} g(x) e^{r\cdot J} - \kappa \cdot J\,, \qquad q = (x, r) \in \mathbb{T}^m \times \mathbb{T}^\ell\,. \tag{1.10}$$

If Y is regarded as a vector field on Λ by identifying w and x with $(w, 0)$ and $(x, 0)$, respectively, then the above relation between g and f can be written as

$$g = \Theta_\star f\,, \qquad \Theta(q, y) = \bigl(q, e^{r\cdot J} y\bigr)\,. \tag{1.11}$$

In order to simplify the discussion, assume now that ω has length one. If $\omega = (w, \kappa)$ is Diophantine of type (1.7) and g belongs to the manifold $\Theta_\star\mathcal{M}$, then the flow for $X = (\omega, f\,.)$ can be trivialized with a change of coordinates V, as described in Theorem 1.1. The same now holds for g. However, the corresponding change of variables $W(q) = V(q)e^{-r\cdot J}$ is not of the desired form, since it still depends on the coordinates r_j. But as we will see,

$$\Phi_Y^t(x) = W(x + t\omega)^{-1} W(x) = V(x + tw)^{-1} e^{tC} V(x)\,, \tag{1.12}$$

for some matrix $C \in \mathfrak{A}$ with frequency vector κ, provided that V is differentiable. What remains to be shown, in specific cases, is that the space of functions of the type (1.10) has a reasonable intersection with the manifold $\mathcal{M}$.

This procedure can be characterized as transforming some circular motion on $\mathfrak{G}$ into motion on an extended torus. Our motivation for this approach is to try to treat all frequencies of the system in a unified way. In the case discussed below, it also has the advantage that the analysis of near-constant skew flows $Y = (\omega, g\,.)$ can be reduced to a purely local analysis near $f \equiv 0$.

Consider now $\mathfrak{G} = \mathrm{SL}(2, \mathbb{R})$. In this case, there is a natural rotation number that can be associated with a skew flow, due to the fact that the fundamental group of $\mathfrak{G}$ is $\mathbb{Z}$ (as for higher dimensional symplectic groups). To this end, consider the flow for $Y = (w, g\,.)$ on the product of $\mathbb{T}^{d-1}$ with $\mathbb{R}^2 \setminus \{0\}$,

$$\dot{v}(t) = g(x_0 + tw)v(t)\,, \qquad v(0) = v_0\,. \tag{1.13}$$

Denote by $\alpha(t)$ the angle between $v(t)$ and some fixed unit vector u_0, and let $\alpha_0 = \alpha(0)$. Then the lift of this angle to $\mathbb{R}$ evolves according to the equation

$$\dot{\alpha}(t) = -\bigl\langle e^{-\alpha(t)J} J g(x_0 + tw) e^{\alpha(t)J} u_0\,, u_0 \bigr\rangle\,, \qquad \alpha(0) = \alpha_0\,, \tag{1.14}$$

where $\langle\,.\,,.\rangle$ denotes the standard inner product on $\mathbb{R}^2$. Here, and in the remaining part of this section, $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. If the components of w are rationally independent, then we can define the so-called *fibered rotation number* of Y,

$$\varrho(Y) = \lim_{t \to \infty} \frac{\alpha(t)}{t}\,. \tag{1.15}$$

As was shown in [23], this limit exists for all $x_0 \in \mathbb{T}^{d-1}$ and $\alpha_0 \in \mathbb{R}$, and it is independent of these initial conditions.

From the definition of Θ, we see that $\varrho(Y) = \kappa$ if and only if $\varrho(X) = 0$. Thus, we may restrict our analysis to skew flows with fibered rotation number zero. Theorem 1.1 deals with precisely such flows. However, the functions (1.10) are of a particular type, and more can be said in this case.

In the following theorem, $\mathfrak{G} = \mathrm{SL}(2, \mathbb{R})$, and $\mathfrak{A}$ is the corresponding Lie algebra of real traceless 2×2 matrices. Denote by $\mathcal{A}_\gamma^0$ the subspace of functions g in $\mathcal{A}_\gamma$ with the property that $g(q) = g(x)$, for all $q = (x, r)$ in $\mathbb{T}^{d-1} \times \mathbb{T}^1$.

Theorem 1.2. *Given $\gamma \geq \gamma_2 > \gamma_0(\beta)$ and $a > 0$, the following holds for some $R > 0$. Consider a constant skew system (w, A) on $\mathbb{T}^{d-1} \times \mathfrak{G}$, for a matrix $A \in \mathfrak{A}$ that has purely imaginary eigenvalues, say $\pm\kappa i$. Assume that $\omega = (w, \kappa)$ satisfies the Diophantine condition (1.7), and that $\|A\| \leq a|\kappa|\|\omega\|$. Then there exists an open neighborhood B_0 of the constant function $x \mapsto A$ in $\mathcal{A}_\gamma^0$, containing a ball of radius R centered at this function, such that for any $g \in B_0$, the one-parameter family $\lambda \mapsto g + \lambda A$ contains a unique member in B_0, say g', whose associated skew flow has a fibered rotation number κ. If $\gamma - \gamma_2 = \varepsilon \geq 1$, then g' is reducible to a constant $C \in \mathfrak{A}$, as described by equation*

(1.12), via a change of coordinates $V \in \mathcal{G}_\varepsilon$. Furthermore, the function g', and (if $\varepsilon \geq 1$) the quantities C and V, depend real analytically on g.

This theorem is proved by first performing a simple change of coordinates $g \mapsto L^{-1}gL$ with $L \in \mathfrak{G}$, such that $L^{-1}AL = \kappa J$, followed by a constant scaling $Y \mapsto cY$ of the resulting skew system, which converts (w, κ) to a unit vector. This is where the condition $\|A\| \leq a|\kappa|\|\omega\|$ comes in. After that, the task is reduced via the map Θ to the study of vector fields $X = (\omega, f\,.)$ with f of the type (1.10). Thus, in view of Theorem 1.1, it suffices to prove (besides real analyticity) that the family $\lambda \mapsto f + \lambda J$ intersects the manifold $\mathcal{M}$ in exactly one point, that $\varrho(X) = 0$ implies $f \in \mathcal{M}$, and that (1.12) holds if f belongs to $\mathcal{M}$.

Our analysis of skew systems near $(\omega, 0)$, including the proof of Theorem 1.1, is based on the use of renormalization group (RG) transformations. These transformations are defined in the next section. As described in more detail in Section 4, each Diophantine vector ω determines, via a multidimensional continued fractions expansion [19], a sequence of matrices $T_n \in \mathrm{SL}(d, \mathbb{Z})$. The n-th step RG transformation $\mathcal{N}_n$ involves a change of variables $(q, y) \mapsto (T_n q, y)$, and another change of variables of the form (1.5), which eliminates certain "nonresonant modes". This is similar in spirit to the RG transformations used in [11-19]. The details of the elimination procedure can be found in Section 3. Each transformation $\mathcal{N}_n$ has $f \equiv 0$ as a fixed point, and the stable/unstable subspaces of $D\mathcal{N}_n(0)$ are the same for all n. Thus, it is possible to define and construct a "stable manifold" (the manifold $\mathcal{M}$ described in Theorem 1.1) for the sequence $\{\mathcal{N}_n\}$. This construction is carried out in Section 5, by extending our RG transformations to parametrized families. The reducibility of functions $f \in \mathcal{M}$ is proved in Section 6, by combining the partial reductions (elimination of nonresonant modes) from the individual RG steps. The remaining results concerning $\mathfrak{G} = \mathrm{SL}(2, \mathbb{R})$ are proved in Section 7.

2. Renormalization

We start by describing a single RG step. A unit vector $\omega \in \mathbb{R}^d$, and a matrix T in $\mathrm{SL}(d, \mathbb{Z})$ are assumed to be given, subject to certain conditions that will be described below. The matrix T defines a map $\mathcal{T} : \Lambda \to \Lambda$,

$$\mathcal{T}(q, y) = \bigl(T(q), y\bigr)\,, \tag{2.1}$$

and the pushforward of a vector field (1.1) under this map is given by

$$\bigl(\mathcal{T}_* X\bigr)(q, y) = \bigl(T\omega, (\mathcal{T}_\star f)(q)y\bigr)\,, \qquad \mathcal{T}_\star f = f \circ T^{-1}\,. \tag{2.2}$$

For every positive $\tau < 1$, define $\mathcal{K}(\tau)$ to be the set of all vectors in $\mathbb{R}^d$ that are contracted by a factor $\leq \tau$ under the action of $S = (T^*)^{-1}$. Here, T^* denotes the transpose of T. Given a fixed value for this contraction factor τ, to be specified later, the "resonant" part $\mathbb{I}^{+}f$ of a function $f \in \mathcal{F}_\gamma$, and its "nonresonant" part $\mathbb{I}^{-}f$, are defined by the equation

$$\mathbb{I}^{\pm} f(q) = \sum_{\nu \in I^{\pm}} f_\nu e^{i\nu\cdot q}\,, \tag{2.3}$$

where $I^+ = \mathcal{K}(\tau) \cap \mathbb{Z}^d$ and $I^- = \mathbb{Z}^d \setminus I^+$. As one would expect (see the lemma below), the resonant part of a function $f \in \mathcal{F}_\gamma$ is contracted under the action of $\mathcal{T}_\star$.

In order to simplify notation, we will drop the subscript γ from now on, unless two different choices of γ are being considered at the same time.

Lemma 2.1. *If $f \in \mathcal{F}$ satisfies $\mathbb{I}^- f = \mathbb{E} f = 0$, then $\|\mathcal{T}_\star f\| \le \tau^\gamma \|f\|$.*

The proof follows immediately from the definitions:

$$\|\mathcal{T}_\star f\| = \sum_{0 \ne \nu \in I^+} \|f_\nu\| (2\|S\nu\|)^\gamma \le \sum_{0 \ne \nu \in I^+} \|f_\nu\| (2\tau\|\nu\|)^\gamma = \tau^\gamma \|f\| \,.$$

The complementary property of the nonresonant modes is that they can easily be eliminated via a change of variables of the form (1.5). To be more precise, we assume that the constant τ can be (and has been) chosen in such a way that $\mathcal{K}(\tau/2)$ contains the orthogonal complement of ω. Under this assumption, we will show in Section 3 that if $f \in \mathcal{F}$ is sufficiently close to zero, then it is possible to find $U_f \in \mathcal{F}$ close to the identity, such that

$$\mathbb{I}^- (\mathcal{U}_f)_\star f = 0 \,. \tag{2.4}$$

By construction, the map $f \mapsto U_f$ is analytic, and U_f belongs to $\mathcal{G}$ whenever $f \in \mathcal{A}$. The renormalized function $\mathcal{N}(f)$ and the renormalized vector field $\mathcal{R}(X)$ are now defined by the equation

$$\mathcal{N}(f) = \eta^{-1} \mathcal{T}_\star (\mathcal{U}_f)_\star f \,, \qquad \mathcal{R}(X) = \eta^{-1} \mathcal{T}_* (\mathcal{U}_f)_* X \,, \tag{2.5}$$

where η is the norm of $T\omega$, so that the torus component of $\mathcal{R}(X)$ is again a unit vector. The corresponding flow is given by

$$\Phi^t_{\mathcal{R}(X)} = \left[U_f(\,.\, + \eta^{-1} t\omega) \Phi_X^{\eta^{-1} t} U_f^{-1} \right] \circ T^{-1} \,. \tag{2.6}$$

In what follows, the RG transformation $\mathcal{N}$ is regarded as a map from an open domain in $\mathcal{F}$ to $\mathcal{F}$. But it should be kept in mind that its restriction to $\mathcal{A}$ takes values in $\mathcal{A}$. An explicit bound on the map $f \mapsto U_f$ leads to the following.

Theorem 2.2. *Let $f = C + h$, with C constant and $\mathbb{E} h = 0$. Assume that $\|C\| < \sigma/6$ and $\|h\| < 2^{-9}\sigma$, with σ satisfying $2\sigma\|S\| < \tau$. Then*

$$\mathcal{N}(f) = \eta^{-1} \left[C + \tilde{h} \right] , \qquad \left\| \tilde{h} \right\| \le \tfrac{3}{2} \tau^\gamma \|h\| \,, \qquad \left| \mathbb{E} \tilde{h} \right| \le 16 \sigma^{-1} \tau^\gamma \|h\|^2 \,. \tag{2.7}$$

$\mathcal{N}$ is analytic on the region determined by the given bounds on C and h. Furthermore, if f is real-valued, then so is $\mathcal{N}(f)$.

A proof of this theorem will be given in Section 3. Notice that the zero-average part h of f gets contracted by roughly a factor τ^γ relative to the constant part C, which is the same factor that appears in Lemma 2.1. The restriction on the size of the domain of $\mathcal{N}$, which is of the order of σ, comes from the solution of equation (2.4).

The goal now is to compose RG transformations of this type, as long as the constant part of f does not become too large. Given a sequence of matrices $P_0, P_1, P_2, \ldots$ in $\mathrm{SL}(d,\mathbb{Z})$, with P_0 the identity, and a unit vector ω_0 in $\mathbb{R}^d$, we define

$$T_n = P_n P_{n-1}^{-1}\,, \quad S_n = (T_n^*)^{-1}\,, \quad \lambda_n = \|P_n\omega_0\|\,, \quad \omega_n = \lambda_n^{-1}P_n\omega_0\,, \tag{2.8}$$

for $n = 1, 2, \ldots$. The following theorem will be proved in Section 4, using as input certain estimates from [19].

Theorem 2.3. *Given $\gamma_1 > \gamma_0(\beta)$, there exist two sequences $n \mapsto \sigma_n$ and $n \mapsto \tau_n$ of positive real numbers less than one, both converging to zero, such that the following holds. If ω_0 is a unit vector in $\mathbb{R}^n$ satisfying the Diophantine condition (1.7), then there exists a sequence $n \mapsto P_n$ of unimodular integer matrices, such that with S_n and λ_n as defined in (2.8),*

$$2\sigma_n\|S_n\| < \tau_n\,, \qquad \|S_n\xi\| \le \frac{\tau_n}{2}\|\xi\|\,, \qquad \lambda_n^{-1}\prod_{j=1}^{n}\bigl(4\tau_j^{\gamma_1}\bigr)\cdot\sigma_1 \le \sigma_{n+1}\,, \tag{2.9}$$

whenever $\omega_{n-1}\cdot\xi = 0$, for every positive integer n.

In order to simplify the discussion, the quantities described in this theorem are considered fixed from now on. We also assume that $\gamma \ge \gamma_1$.

The n-th step RG transformation $\mathcal{N}_n$ and the composed RG transformation $\widetilde{\mathcal{N}}_n$ are defined by the equation

$$\mathcal{N}_n(f) = \eta_n^{-1}(\mathcal{T}_n)_\star(\mathcal{U}_f)_\star f\,, \qquad \widetilde{\mathcal{N}}_n = \mathcal{N}_n \circ \mathcal{N}_{n-1} \circ \ldots \circ \mathcal{N}_1\,, \tag{2.10}$$

where $\eta_n = \lambda_n/\lambda_{n-1}$ for $n \ge 1$, with $\lambda_0 = 1$. To be more specific, we choose $\tau = \tau_n$ and $\omega = \omega_{n-1}$ in the construction of the map $\mathcal{U}_f$ that enters the definition of $\mathcal{N} = \mathcal{N}_n$.

By Theorem 2.2, the transformation $\mathcal{N}_n$ is well defined on the open ball $B_n \subset \mathcal{F}$ of radius $2^{-9}\sigma_n$, centered at the origin. B_n will be referred to as the domain of $\mathcal{N}_n$. The domain of $\widetilde{\mathcal{N}}_n$ is defined recursively as the set of all functions in the domain of $\widetilde{\mathcal{N}}_{n-1}$ that are mapped into B_n by $\widetilde{\mathcal{N}}_{n-1}$. For such a function f, define $f_0 = f$ and

$$f_n = \widetilde{\mathcal{N}}_n(f_0)\,, \qquad \bar{f}_n = \mathbb{E}f_n\,, \qquad h_n = f_n - \bar{f}_n\,. \tag{2.11}$$

By Theorem 2.2 and Theorem 2.3, we have

$$\|h_n\| \le \lambda_n^{-1}\prod_{j=1}^{n}\bigl(2\tau_j^{\gamma}\bigr)\cdot\|f_0\| \le 2^{-10}\sigma_{n+1}\,. \tag{2.12}$$

This shows e.g. that for $f \in \mathcal{F}$ close to zero, the question of whether or not f is infinitely renormalizable depends only on the size of the averages $\bar{f}_n$. Consider now a sequence ρ of real numbers satisfying

$$0 < \rho_n \le 2^{-10}\sigma_{n+1}\,, \qquad n = 0, 1, 2, \ldots\,. \tag{2.13}$$

Given an open set $B(\gamma) \subset B_1$ containing zero, define $\widetilde{B}_0 = B(\gamma)$ and

$$\widetilde{B}_{n+1} = \left\{ f \in \widetilde{B}_n : \|\bar{f}_n\| < \rho_n \right\}, \qquad n = 0, 1, 2, \ldots . \tag{2.14}$$

The bound (2.12) shows that $\widetilde{B}_{n+1}$ is contained in the domain of $\widetilde{\mathcal{N}}_{n+1}$.

Theorem 2.4. *If $\gamma > \gamma_1$ then there exists a sequence ρ satisfying (2.13), and a non-empty open neighborhood $B(\gamma)$ of the origin in $\mathcal{F}$, such that $\mathcal{M}_\gamma = \bigcap_{n=0}^\infty \widetilde{B}_n$ is the graph of an analytic function $M : (\mathbb{I} - \mathbb{E})B(\gamma) \to \mathbb{E}B(\gamma)$. Both M and its derivative vanish at the origin.*

A proof of this theorem is given in Section 5. The reducibility of functions f belonging to $\mathcal{M} = \mathcal{M}_\gamma$ will be proved in Section 6, by iterating the identity (2.6), and using that $f_n \to 0$, in order to estimate the product of the matrices U_{f_n}.

3. Elimination of nonresonant modes

Here we solve equation (2.4) and prove Theorem 2.2. A unit vector $\omega \in \mathbb{R}^d$ and a matrix T in $\mathrm{SL}(d, \mathbb{Z})$ are assumed to be given. As mentioned in the last section, we also assume that the cone $\mathcal{K}(\tau/2)$ contains the orthogonal complement of ω, and that $2\sigma\|S\| < \tau$.

Proposition 3.1. *If ν belongs to I^- then $|\omega \cdot \nu| > \sigma$.*

Proof. Given $\nu \in I^-$, consider its decomposition $\nu = \nu_\| + \nu_\perp$ into a vector $\nu_\|$ parallel to ω and a vector $\nu_\perp$ perpendicular to ω. By using that $\|\nu\|, \|S\|, \|\omega\| \geq 1$, we obtain

$$\begin{aligned} \sigma \leq \sigma\|\nu\| &< \|S\|^{-1}\frac{\tau}{2}\|\nu\| \leq \|S\|^{-1}\bigl(\|S\nu\| - \|S\nu_\perp\|\bigr) \\ &\leq \|S\|^{-1}\|S\nu_\|\| \leq \|\nu_\|\| \leq |\omega \cdot \nu|, \end{aligned}$$

as claimed. **QED**

Given any $n \times n$ matrix C, define $\hat{C}f = fC - Cf$ for every function $f \in \mathcal{F}$.

Proposition 3.2. *Assume that $\|C\| \leq \sigma/4$. Then the linear operators $D_\omega = \omega \cdot \nabla$ and $\mathcal{D} = D_\omega + \hat{C}$ commute with $\mathbb{I}^-$, have bounded inverses when restricted to $\mathbb{I}^-\mathcal{F}$, and satisfy*

$$\bigl\|D_\omega^{-1}\mathbb{I}^-\bigr\| \leq \sigma^{-1}, \qquad \bigl\|D_\omega\mathcal{D}^{-1}\mathbb{I}^-\bigr\| \leq 2. \tag{3.1}$$

Proof. Clearly, D_ω, $\hat{C}$, and $\mathbb{I}^-$ commute with each other. The first inequality in (3.1) follows immediately from Proposition 3.1. It implies $\|D_\omega^{-1}\hat{C}\mathbb{I}^-\| \leq 2\sigma^{-1}\|C\| \leq 1/2$, and the indicated bound on $D_\omega\mathcal{D}^{-1}\mathbb{I}^- = (\mathbb{I} + D_\omega^{-1}\hat{C})^{-1}\mathbb{I}^-$ is now obtained via Neumann series. **QED**

In the rest of this paper, we will frequently use analyticity arguments. Thus, let us recall at this point some relevant facts [21] about

analytic maps. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{C}$, and let $B \subset \mathcal{X}$ be open. We say that $G : B \to \mathcal{Y}$ is analytic if it is Fréchet differentiable. Thus, sums, products, and compositions of analytic maps are analytic. Equivalently, G is analytic if it is locally bounded, and if for all continuous linear maps $f : \mathbb{C} \to \mathcal{X}$ and $h : \mathcal{Y} \to \mathbb{C}$, the function $h \circ G \circ f$ is analytic. This shows e.g. that uniform limits of analytic functions are analytic. Assuming that B is a ball of radius r and that F is bounded on B, a third equivalent condition is that G has derivatives of all orders at the center of B, and that the corresponding Taylor series has a radius of convergence at least r and agrees with G on B.

Another fact that we will use repeatedly is that $\mathcal{F}$ is a Banach algebra, i.e., we have $\|fg\| \le \|f\|\|g\|$ for all $f, g \in \mathcal{F}$.

In the remaining part of this section, $f \in \mathcal{F}$ is fixed but arbitrary, $C = \mathbb{E}f$, and $h = f - C$. We seek a solution of equation (2.4) of the form $U = \exp(\mathcal{D}^{-1}u)$, with u a function in $\mathbb{I}^{-}\mathcal{F}$. In order to simplify notation, $\mathbb{E}\mathcal{F}$ will be identified with $\mathrm{GL}(n, \mathbb{C})$. A short computation shows that

$$\mathbb{I}^{-}\mathcal{U}_{\star} f = u - \psi(u)\,, \tag{3.2}$$

where

$$\begin{aligned}\psi(u) = -\mathbb{I}^{-}\big[&(D_\omega \mathcal{D}^{-1}u)E_1^- + (D_\omega E_2^+)E_0^- + E_0^+ h E_0^- \\ &+ CE_2^- + (\mathcal{D}^{-1}u)CE_1^- + E_2^+ C E_0^-\big]\end{aligned} \tag{3.3}$$

and

$$E_m^{\pm} = \sum_{k=m}^{\infty} \frac{1}{k!}\big(\pm\mathcal{D}^{-1}u\big)^k\,, \qquad m = 0, 1, \ldots\,. \tag{3.4}$$

Proposition 3.3. *Assume that $\|C\| < \sigma/6$ and $\|h\| < 2^{-9}\sigma$. Let $r = 2^{-8}\sigma$, and denote by B_r the closed ball of radius r in $\mathbb{I}^{-}\mathcal{F}$, centered at the origin. Then ψ has a unique fixed point u_f in B_r, and*

$$\|u_f\| \le \tfrac{16}{15}\|h\|\,. \tag{3.5}$$

The map $(C, h) \mapsto u_f$ is analytic on the domain defined by the given bounds on C and h. If f is real-valued, then so is u_f. Furthermore, if f belongs to $\mathcal{A}$ then $U_f = \exp(\mathcal{D}^{-1}u_f)$ belongs to $\mathfrak{G}$.

Proof. First, recall that $e^x \le (1-x)^{-1}$ whenever $0 \le x < 1$. This fact will be used below and in subsequent proofs.

A straightforward estimate, using Proposition 3.2 and the Banach algebra property of $\mathcal{F}$, shows that ψ is an analytic map from the space $\mathbb{I}^{-}\mathcal{F}$ to itself, satisfying the bound

$$\big\|\psi(u)\big\| \le e^{4\sigma^{-1}\|u\|}\big(\|h\| + 10\sigma^{-1}\|u\|^2\big)\,. \tag{3.6}$$

Notice that $\psi(0) = -\mathbb{I}^{-}h$ has norm $\le r/2$. Thus, if we prove that $\|D\psi(u)\| \le 1/2$ for all $u \in B_r$, then the existence and uniqueness of a fixed point $u_f \in B_r$ follows from the contraction mapping principle.

Let $u \in B_r$ and $g \in \mathcal{F}$ be fixed but arbitrary, with $\|g\| = 1$. Define $\varphi : \mathbb{C} \to \mathcal{F}$ by the equation $\varphi(z) = \psi(u + zg)$. If $|z| \le R = 2^{-6}\sigma$, then $u + zg$ is bounded in norm by $\sigma/48$, and by using (3.6), we find that

$$\|\varphi(z)\| \le \tfrac{12}{11}\|h\| + 11\sigma^{-1}\|u + zg\|^2 < R/2 . \tag{3.7}$$

Thus, by Cauchy's formula,

$$\|D\psi(u)g\| = \|\varphi'(0)\| \le R^{-1} \sup_{|z|=R} \|\varphi(z)\| < 1/2 . \tag{3.8}$$

As mentioned above, this proves the existence and uniqueness of the fixed point u_f in B_r. By equation (3.6), this fixed point satisfies

$$\|u_f\| \le \tfrac{64}{63}\bigl(\|h\| + \tfrac{1}{23}\|u_f\|\bigr) , \tag{3.9}$$

which implies the bound (3.5).

The analyticity of the map $(C, h) \mapsto u_f$ follows from the uniform convergence of the series (3.4) and of the sequence $\psi^n(0) \to u_f$, together with the chain rule. If f is real-valued, then the equation (3.3) shows that $\psi^n(0)$, and thus u_f as well, is real-valued. Similarly, if f belongs to $\mathcal{A}$ then so does u_f, implying that $U_f \in \mathcal{G}$. **QED**

For reference later on, we note that Proposition 3.3 and Proposition 3.2 imply the bound

$$\|U_f - \mathrm{I}\| \le \exp\bigl(3\sigma^{-1}\|(\mathbb{I} - \mathbb{E})f\|\bigr) - 1 . \tag{3.10}$$

Lemma 3.4. *If $f = C + h$, with $\|C\| \le \sigma/6$ and $\|h\| \le 2^{-9}\sigma$, then*

$$\bigl\|(\mathcal{U}_f)_\star f - \overset{+}{\mathbb{I}} f\bigr\| \le 16\sigma^{-1}\|h\|^2 . \tag{3.11}$$

Proof. An explicit computation shows that

$$\begin{aligned}(\mathcal{U}_f)_\star f - \overset{+}{\mathbb{I}} f = \overset{+}{\mathbb{I}}\bigl[&(D_\omega \mathcal{D}^{-1}u)E_1^- + (D_\omega E_2^+)E_0^- + hE_1^- + E_1^+ h E_0^- \\ &+ CE_2^- + (\mathcal{D}^{-1}u)CE_1^- + E_2^+ C E_0^-\bigr] .\end{aligned} \tag{3.12}$$

Using Proposition 3.2 and the Banach algebra property of $\mathcal{F}$, we find that

$$\bigl\|(\mathcal{U}_f)_\star f - \overset{+}{\mathbb{I}} f\bigr\| \le e^{4\sigma^{-1}\|u_f\|}\bigl(4\sigma^{-1}\|u_f\|\|h\| + 10\sigma^{-1}\|u_f\|^2\bigr) . \tag{3.13}$$

The estimate (3.11) is now obtained by substituting the bound on $\|u_f\|$ from Proposition 3.3. **QED**

Proof of Theorem 2.2. Using the definition (2.5) of $\mathcal{N}$, the function $\tilde{h}$ in equation (2.7) is given by

$$\tilde{h} = \mathcal{T}_\star\bigl[\overset{+}{\mathbb{I}} h + (\mathcal{U}_f)_\star f - \overset{+}{\mathbb{I}} f\bigr] .$$

The given bounds in (2.7) now follow from Lemma 3.4 and Lemma 2.1. In particular, we have

$$\|\tilde{h}\| \le \tau^{\gamma}\bigl(\|h\| + 16\sigma^{-1}\|h\|^2\bigr) \le \tfrac{33}{32}\tau^{\gamma}\|h\|\,, \tag{3.14}$$

as claimed. The analyticity of $\mathcal{N}$ follows from the analyticity of the map $f \mapsto u_f$, the uniform convergence of (3.4), and the chain rule. If f is real-valued, then so is u_f by Proposition 3.3, and thus $\mathcal{N}(f)$ is real-valued as well. **QED**

The following facts about torus-translations will be used later on. If f is a function on $\mathbb{T}^d$ and p a point on this torus, define $(R_p f)(q) = f(q+p)$ for all $q \in \mathbb{T}^d$. These translation operators R_p commute with the projections $\mathbb{I}^{\pm}$ defined in (2.3). As a result, they also commute with $f \mapsto U_f$, as can be seen from our construction of this map. A straightforward computation now shows that

$$\mathcal{N} \circ R_p = R_{Tp} \circ \mathcal{N}\,, \qquad p \in \mathbb{T}^d\,. \tag{3.15}$$

4. Choice of integer matrices

We give a brief description of the the multidimensional continued fractions expansion of [19], which is based on the work of [24,25] on geodesic flows on homogeneous spaces. Then we use the estimates from [19] on the resulting integer matrices P_n to prove Theorem 2.3.

Let F be a fundamental domain for the left action of $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ on $G = \mathrm{SL}(d,\mathbb{R})$. Consider the one-parameter subgroup of G, generated by the matrices

$$E^t = \mathrm{diag}\bigl(e^{-t},\ldots,e^{-t},e^{(d-1)t}\bigr)\,, \qquad t \in \mathbb{R}\,, \tag{4.1}$$

and the corresponding flow on the quotient space $\Gamma\backslash G$, defined by $\Gamma W \mapsto \Gamma W E^t$. Given a vector $\omega \in \mathbb{R}^d$ of the form $\omega = (w,1)$, define $W \in G$ to be the matrix obtained from the $d \times d$ identity matrix by replacing its last column vector by ω. Then, for every $t \in \mathbb{R}$, there exists a unique matrix $P(t) \in \Gamma$ such that $P(t)WE^t$ belongs to F. To a given sequence of "stopping times" $t_n \ge 0$ we can now associate a sequence of matrices $P_n = P(t_n)$. The corresponding matrices T_n and S_n are defined as in (2.8).

Let $\theta = \beta/(d+\beta)$.

Theorem 4.1. *([19]) There are constants $c_1, c_2, c_3 > 0$, such that the following holds. If $\omega = (w,1)$ is any vector of length less than d, satisfying the Diophantine condition (1.7), and if $n \mapsto t_n$ is any sequence of stopping times, with $t_0 = 0$ and $\delta t_n = t_n - t_{n-1} > 0$, then the bounds*

$$\begin{aligned} \|P_n^{-1}\| &\le c_1 \exp\{(d-1+\theta)t_n\}, \\ \|S_n\| &\le c_2 \exp\{(d-1)(1-\theta)\delta t_n + d\,\theta\, t_n\}, \\ \|S_n\xi\| &\le c_3 \exp\{-(1-\theta)\delta t_n + d\,\theta t_{n-1}\}\,, \end{aligned} \tag{4.2}$$

hold for all integers $n > 0$, and for all unit vectors $\xi \in \mathbb{R}^d$ that are perpendicular to $P_{n-1}\omega$.

Remark. The condition $\|\omega\| < d$ was added in this theorem to have constants c_k that do not depend on the length of ω. For an arbitrary Diophantine $\omega \in \mathbb{R}^d$, it is always possible to permute basis vectors in $\mathbb{R}^d$, and to rescale ω, in such a way that the last component is equal to 1 and $\|\omega\|$ less than d.

Define now

$$\sigma_n = \sigma_0 e^{-d\delta t_n}\,, \qquad \tau_n = \tau_0 \exp\{-[(d-1)\theta+1]\delta t_n + d\theta t_n\}\,, \tag{4.3}$$

with $\sigma_0 = \tau_0/(3c_2)$ and $\tau_0 = 2c_3$. Then the first two inequalities in (2.9) are an immediate consequence of the last two bounds in (4.2). In addition, we have

$$\tau_1\tau_2\cdots\tau_n \le \tau_0^n \exp\{-(1-\theta)t_n + d\,\theta s_{n-1}\}\,, \tag{4.4}$$

where $s_k = t_1 + t_2 + \ldots t_k$. Define $\mu = (\gamma+1)(1-\theta) - d$. Consider the constants λ_n defined in (2.8). By using the trivial estimate $\lambda_n^{-1} \le \|P_n^{-1}\|$, together with the first bound in (4.2), we obtain

$$\lambda_n^{-1} \prod_{k=1}^{n} \bigl(4\tau_k^\gamma\bigr)\sigma_0\sigma_{n+1}^{-1} \le c_1 4^n \tau_0^{\gamma n} \exp\{-\mu t_n + \gamma d\,\theta s_{n-1} + d\,\delta t_{n+1}\}\,. \tag{4.5}$$

This implies the third inequality in (2.9), provided that the sequence $\{t_n\}$ can be chosen in such a way that the right hand side of (4.5) is less than 1. To this end, let

$$t_n = c(1+\alpha)^n\,, \qquad n = 1, 2, \ldots\,,$$

with $c, \alpha > 0$ to be determined. By using that $\delta t_{n+1} = \alpha t_n$ and $s_{n-1} \le \alpha^{-1} t_n$, we find that

$$\lambda_n^{-1} \prod_{k=1}^{n} \bigl(4\tau_k^\gamma\bigr)\sigma_0\sigma_{n+1}^{-1} \le c_1 4^n \tau_0^{\gamma n} \exp\{-\varepsilon c(1+\alpha)^n\}\,, \tag{4.6}$$

where $\varepsilon = \mu - \gamma d\,\theta\alpha^{-1} - d\,\alpha$. The goal is to choose α in such a way that $\varepsilon > 0$. Then by taking $c > 0$ sufficiently large, the right hand side of (4.6) is less than one, for all positive integers n, and the third bound in (2.9) follows. The condition $\varepsilon > 0$ is a quadratic inequality for α, which is satisfied by $\alpha = \mu/(2d)$, provided that $\mu^2 > 4\gamma\theta d^2$. An explicit computation shows that μ^2 is larger than $4\gamma\theta d^2$, whenever

$$\gamma > \frac{d}{(1-\theta)^2}\Bigl[(1-\theta+2d\,\theta) + 2\sqrt{\theta}\sqrt{d^2-(1-\theta)(d^2-d+1-\theta)}\,\Bigr] - 1\,. \tag{4.7}$$

The same condition also guarantees that μ, and thus α, is positive. Substituting $\theta = \beta/(d+\beta)$ into the inequality (4.7), one gets the equivalent condition $\gamma > \gamma_0(\beta)$, with $\gamma_0(\beta)$ as defined in equation (1.9). Finally, substituting the bound (4.7) on γ into the definition of μ yields $\alpha = \mu/(2d) > d\theta/(1-\theta)$, which shows that $\tau_{n+1} = \tau_0 \exp\{[-(1-\theta)\alpha + d\theta]t_n\}$ tends to zero as $n \to \infty$. Taking $c > 0$ sufficiently large ensures that $\tau_n < 1$ for all n. The analogous property $1 > \sigma_n \to 0$ follows now from the first inequality in (2.9). This completes the proof of Theorem 2.3.

5. The stable manifold

In this section, we define RG transformations for families of functions in $\mathcal{A}$, parametrized by $\mathfrak{A}$. These transformations $\mathfrak{N}_n$ are then used to prove Theorem 2.4 and some other estimates that are needed later on. $\mathfrak{N}_n$ acts on a family $F : \mathfrak{A} \to \mathcal{A}$ by composing it from the left with $\mathcal{N}_n$, and from the right with a reparametrization map on $\mathfrak{A}$ that depends on F. Recall that $\mathcal{N}_n$ is naturally defined on an open domain in $\mathcal{F}$, but that its restriction to $\mathcal{A}$ takes values in $\mathcal{A}$. The situation is analogous for $\mathfrak{N}_n$, since, as will be clear from the construction, the reparametrization map takes real values for real arguments, whenever F is real. Thus, no generality is lost by assuming that $\mathfrak{G} = \mathrm{GL}(n, \mathbb{C})$. We will do this in the remaining part of this paper, unless specified otherwise.

We start with a preliminary estimate on inverses of some simple maps. Denote by b the open unit ball in $\mathfrak{A}$, centered at the origin. Consider the space $\mathcal{U}$ of analytic functions $U : b \to \mathfrak{A}$, equipped with the sup-norm.

Proposition 5.1. *Let $0 < \lambda < \frac{1}{3}$, and let $U \in \mathcal{U}$ with $\|U\| < \frac{1}{2}$. Define $\Lambda(A) = \lambda A$ for every $A \in \mathfrak{A}$. Then $\Lambda^{-1} + U$ has a unique right inverse $\Lambda + V$ on b, with V belonging to $\mathcal{U}$ and satisfying $\|V\| \le \lambda\|U\|$. The map $U \mapsto V$ is analytic on the domain in $\mathcal{U}$ defined by the given condition on U.*

Proof. If A is a matrix in $\mathfrak{A}$ of norm less than $2/3$, and C a matrix in $\mathfrak{A}$ of norm one, then from Cauchy's formula, we obtain

$$\|DU(A)C\| \le 3 \sup_{|z|=1/3} \|U(A + zC)\| \le 3\|U\| < 3/2. \tag{5.1}$$

Now consider the equation for V, which can be written as $\psi(V) = V$, with ψ defined by $\psi(V) = -\lambda U \circ (\Lambda + V)$. Denote by B the closed ball of radius $r = 1/3$ in $\mathcal{U}$, centered at the origin. Then ψ is analytic on B, with derivative given by

$$D\psi(V)H = -\lambda\big((DU) \circ (\Lambda + V)\big)H\,. \tag{5.2}$$

By equation (5.1), we see that $\|D\psi(V)\| < 1/2$, for all $V \in B$. Since $\|\psi(0)\| \le r/2$, the map ψ is a contraction on B, and thus has a (unique) fixed point in B. This fixed point V satisfies $\|V\| = \|\psi(V)\| \le \lambda\|U\|$. The analyticity of $U \mapsto V$ follows form the uniform convergence of $\psi^n(0) \to V$ for $\|U\| < 1/2$. **QED**

Next, let $\rho_0 = 2^{-11}\sigma_1$ and

$$\rho_n = \lambda_n^{-1} 4^{-n} \pi_n^{\gamma_1} \rho_0\,, \qquad \pi_n = \prod_{j=1}^{n} \tau_j\,, \qquad n = 1, 2, \ldots\,. \tag{5.3}$$

For every integer $n \ge 0$, define $\mathfrak{A}_n$ to be the vector space $\mathfrak{A}$, equipped with the norm $\|s\|_n = \rho_n^{-1}\|s\|$. Denote by b_n the open unit ball in $\mathfrak{A}_n$, centered at the origin. Define $\mathcal{B}_n$ to be the space of analytic families $F : b_n \to \mathcal{A}$, equipped with the norm

$$\|F\|_n = \sup_{s \in b_n} \|F(s)\|_n\,. \tag{5.4}$$

The inclusion map from $\mathfrak{A}_n$ into $\mathbb{E}\mathcal{A}$ will be denoted by F^0. In other words, $F^0(s) = s$. Notice that F^0 has norm one in $\mathcal{B}_n$.

Let $n \geq 1$. By Theorem 2.3, we have $\rho_{n-1} < 2^{-4n-6}\sigma_n$. Thus, if $\|F\|_{n-1} < 2^{4n-3}$, then $F(s)$ belongs to the domain of $\mathcal{N}_n$, for all $s \in b_n$. We can associate to each such F an analytic map

$$Y_{n,F} = \mathbb{E}(\mathcal{N}_n \circ F) \tag{5.5}$$

from b_{n-1} to $\mathfrak{A}_n$. Notice that, by Theorem 2.2, if F takes real values for real arguments, then so does $Y_{n,F}$. On the space of analytic maps $b_{n-1} \to \mathfrak{A}$, we will use the topology of uniform convergence (sup-norm).

Proposition 5.2. *Assume that $F \in \mathcal{B}_{n-1}$ satisfies $\|F - F^0\|_{n-1} < 1$ and $\mathbb{E}F = F^0$. Then $Y_{n,F} : b_{n-1} \to \mathfrak{A}_n$ has a unique right inverse $Y_{n,F}^{-1} : b_n \to b_{n-1}$. Both $Y_{n,F}$ and its right inverse depend analytically on F, on the domain defined by the given condition on F. Furthermore,*

$$\begin{aligned}
\|Y_{n,F}(s) - \eta_n^{-1}s\|_n &\leq 2^{-4n}\epsilon\tau_n^{\gamma-\gamma_1}\|F - F^0\|_{n-1}\,, && s \in b_{n-1}\,,\\
\|DY_{n,F}(s) - \eta^{-1}\mathrm{I}\| &\leq 2^{-4n-6}\epsilon\tau_n^{\gamma-\gamma_1}\|F - F^0\|_{n-1}\,, && s \in b_{n-1}\,,\\
\|Y_{n,F}^{-1}(s) - \eta_n s\|_{n-1} &\leq 2^{-4n-1}\epsilon\tau_n^{\gamma}\|F - F^0\|_{n-1}\,, && s \in b_n\,.
\end{aligned} \tag{5.6}$$

Here, I *denotes the inclusion map from $\mathfrak{A}_{n-1}$ into $\mathfrak{A}_n$, and $\epsilon = 2^{4n+6}\sigma_n^{-1}\|F - F^0\| < 1$.*

Proof. By Theorem 2.2, the map $Y = Y_{n,F}$ satisfies the bound

$$\begin{aligned}
\|\eta_n Y(s) - s\| &= \|\eta_n \mathbb{E}\mathcal{N}_n(F(s)) - s\| \leq \|\mathbb{E}F(s) - s\| + 16\sigma_n^{-1}\tau_n^{\gamma}\|(\mathbb{I} - \mathbb{E})F(s)\|^2\\
&= 16\sigma_n^{-1}\tau_n^{\gamma}\|F(s) - s\|^2 \leq 2^{-4n-2}\epsilon\tau_n^{\gamma}\|F - F^0\|\\
&= 2^{-4n}\epsilon\tau_n^{\gamma-\gamma_1}\eta_n\rho_n\|F - F^0\|_{n-1}\,,
\end{aligned} \tag{5.7}$$

with ϵ as defined above, for all $s \in b_{n-1}$. Dividing both sides by $\eta_n\rho_n$ yields the first inequality in (5.6). The inequality $\epsilon < 1$ follows from the fact that $\|F - F^0\| \leq \rho_{n-1} < 2^{-4n-6}\sigma_n$.

Consider $C \in \mathfrak{A}$ of norm one, and $z \in \mathbb{C}$ of absolute value $\leq 2^6$. Given the allowed size of $\|C\|$ in Theorem 2.2, the bound (5.7) still holds if s is replaced by $s + zC$. Thus, the second inequality in (5.6) is obtained from the first, using a Cauchy estimate with contour $|z| = 2^6$.

In order to simplify notation, if λ is a scalar, then the map $s \mapsto \lambda s$ will be denoted by λ as well. Let now $\lambda = \eta_n\rho_n/\rho_{n-1}$. Consider the space $\mathcal{U}$ introduced before Proposition 5.1. Then $U = \rho_n^{-1}(Y - \eta_n^{-1})\rho_{n-1}$ belongs to $\mathcal{U}$. By using the first inequality in (5.6), we obtain

$$\|U(s)\| \leq \rho_n^{-1}\|(Y - \eta_n^{-1})(\rho_{n-1}s)\| \leq 2^{-4}\,, \tag{5.8}$$

for all $s \in \mathfrak{A}$ of norm less than 1. Notice also that $\lambda = \tau_n^{\gamma_1}/4 \leq 1/4$. Thus, Proposition 5.1 guarantees the existence of a unique right inverse $\lambda + V$ for $\lambda^{-1} + U$, with V belonging to $\mathcal{U}$. This yields the right inverse $Y^{-1}=\eta_n+\rho_{n-1}V\rho_n^{-1}$ for Y on b_n. The bound on V from Proposition 5.1, together with the first inequality in (5.6), implies that

$$\begin{aligned}\|Y^{-1}(s)-\eta_n s\| = \rho_{n-1}\|V(\rho_n^{-1}s)\| \le 2\eta_n\rho_n\|U\| \le 2\eta_n\rho_n\|Y-\eta_n^{-1}\|_n \\ \le 2^{-4n-1}\epsilon\tau_n^{\gamma}\rho_{n-1}\|F-F^0\|_{n-1}\,,\end{aligned}\tag{5.9}$$

for all $s\in b_n$. Dividing both sides by ρ_{n-1} yields the third inequality in (5.6).

The analytic dependence on F, of the function $Y_{n,F}$ and its right inverse, follows from Theorem 2.2, Proposition 5.1, and the chain rule. **QED**

This proposition allows us to define the n-th step RG transformation $\mathfrak{R}_n$ and the composed RG transformation $\widetilde{\mathfrak{R}}_n$ for families by

$$\mathfrak{R}_n(F)=\mathcal{N}_n\circ F\circ Y_{n,F}^{-1}\,,\qquad \widetilde{\mathfrak{R}}_n=\mathfrak{R}_n\circ\mathfrak{R}_{n-1}\circ\ldots\circ\mathfrak{R}_1\,.\tag{5.10}$$

Notice that $\mathbb{E}\mathfrak{R}_n(F)=F^0$. In particular, since $\mathcal{N}_n$ maps constant functions to constant functions, F^0 is a fixed point for $\mathfrak{R}_n$. The domain of $\mathfrak{R}_n$ is the set of all $F\in\mathcal{B}_{n-1}$ satisfying $\|F-F^0\|_{n-1}<1$ and $\mathbb{E}F=F^0$. Clearly, $\mathfrak{R}_n$ is analytic on this domain.

In what follows, we assume that $\gamma\ge\gamma_2>\gamma_1$. Let $K\le 1$ be a fixed positive real number satisfying

$$8^n\pi_n^{\gamma_2-\gamma_1}K\le 1/16\,,\tag{5.11}$$

for all integers $n\ge 0$. Such a number K exists by Theorem 2.3.

Lemma 5.3. *If $F_0\in\mathcal{B}_0$ satisfies $\|F_0-F^0\|_0<K$ and $\mathbb{E}F_0=F^0$, then $\widetilde{\mathfrak{R}}_n(F_0)$ is well defined for all $n\ge 1$, and satisfies*

$$\bigl\|\widetilde{\mathfrak{R}}_n(F_0)-F^0\bigr\|_n\le 8^n\pi_n^{\gamma-\gamma_1}\|F_0-F^0\|_0\,.\tag{5.12}$$

Proof. Let $m\ge 1$, and let F be an arbitrary family in the domain of $\mathfrak{R}_m$. Fix $s\in b_m$, and define $s'=Y_{m,F}^{-1}(s)$. By Theorem 2.2 and Proposition 5.2, we have

$$\begin{aligned}\rho_m^{-1}\|\mathfrak{R}_m(F)(s)-F^0(s)\| = \rho_m^{-1}\|(\mathbb{I}-\mathbb{E})\mathcal{N}_m(F(s'))\| \le 2\tau_m^{\gamma}\eta_m^{-1}\rho_m^{-1}\|(\mathbb{I}-\mathbb{E})F(s')\| \\ = 8\tau_m^{\gamma-\gamma_1}\rho_{m-1}^{-1}\|F(s')-F^0(s')\|\,.\end{aligned}\tag{5.13}$$

Consider now F_0 in the domain of $\mathfrak{R}_1$, and assume that the claim of Lemma 5.3 holds for all $n<m$. By setting $F=\widetilde{\mathfrak{R}}_{m-1}(F_0)$ in inequality (5.13), we obtain

$$\bigl\|\widetilde{\mathfrak{R}}_m(F_0)-F^0\bigr\|_m\le 8\tau_m^{\gamma-\gamma_1}\bigl\|\widetilde{\mathfrak{R}}_{m-1}(F_0)-F^0\bigr\|_{m-1}\le 8^m\pi_m^{\gamma-\gamma_1}\|F-F^0\|_0\,.\tag{5.14}$$

This proves (5.12) for $n=m$. Under the given assumptions on F_0, the right hand side of this inequality is less than 1, which shows that $\widetilde{\mathfrak{R}}_m(F_0)$ belongs to the domain of $\mathfrak{R}_{m+1}$. **QED**

In what follows, the set of families satisfying the assumptions of Lemma 5.3 will be referred to as the "domain of $\widetilde{\mathfrak{R}}$". If F_0 is any family in this domain, define

$$F_n=\widetilde{\mathfrak{R}}_n(F_0)\,,\qquad Y_n=Y_{n,F_{n-1}}\,,\qquad Z_{m,n}=Y_{m+1}^{-1}\circ\ldots\circ Y_{n-1}^{-1}\circ Y_n^{-1}\,,\tag{5.15}$$

for all integers $0 \le m < n$.

Proposition 5.4. *Suppose that F belongs to the domain of $\widetilde{\mathfrak{N}}$. Then there exists a unique sequence $m \mapsto z_m \in b_m$ satisfying*

$$z_{m-1} = Y_m^{-1}(z_m)\,, \qquad m = 1, 2, \ldots\,, \tag{5.16}$$

and this sequence is given by the limits $z_m = \lim_{n\to\infty} Z_{m,n}(0)$. The maps $F \mapsto z_m$ are analytic on the domain of $\widetilde{\mathfrak{N}}$. Furthermore, if F takes real values for real arguments, then z_m is real.

Proof. Let $F_0 = F$. We start by establishing a contraction property for Y_n^{-1}. Let $s \in b_n$. By using Proposition 5.2, and the fact that $\|\eta_n s\|_{n-1} = \frac{1}{4}\tau_n^{\gamma_1}\|s\|_n$, we obtain

$$\|Y_n^{-1}(s)\|_{n-1} \le \|\eta_n s\|_{n-1} + \|Y_n^{-1}(s) - \eta_n s\|_{n-1} \le 9/32\,. \tag{5.17}$$

Thus, Y_n^{-1} maps b_n into $b_{n-1}/3$. Furthermore, by Cauchy's formula, the derivative of Y_n^{-1} on the closure of $b_n/3$ is bounded in norm (as an operator from $\mathfrak{A}_n$ to $\mathfrak{A}_{n-1}$) by $1/2$.

Consider now an arbitrary sequence $n \mapsto s_n \in b_n$, with the property that s_n belongs to the closure of $b_n/3$ for $n \ge 1$. Notice that if a sequence $n \mapsto z_n \in b_n$ satisfies (5.16), then it automatically has this property. Define $s_{m,n} = Z_{m,n}(s_n)$ for all integers $0 \le m < n$. By the contraction property of the maps Y_i^{-1}, we have $\|s_{m,k} - s_{m,n}\|_n < 2^{m-n}$ whenever $1 \le m < n < k$. This shows that $n \mapsto s_{m,n}$ converges as $n \to \infty$, and that the limit $\hat{s}_m$ is independent of the sequence $\{s_n\}$. In particular, we see that $\hat{s}_m = z_m$ by choosing $s_n = 0$ for all n. The identities (5.16) are obtained by choosing $s_n = z_n$ for all n.

By Proposition 5.2, the maps $F \mapsto s_{m,n} = Z_{m,n}(0)$ are analytic on the domain of $\widetilde{\mathfrak{N}}$. The analyticity of $F \mapsto z_m$ now follows from the uniform convergence of $s_{m,n} \to z_m$. If F_{n-1} is real (takes real values for real arguments) for some $n > 0$, then so is Y_n, as mentioned earlier, and thus also Y_n^{-1} and F_n. By induction, we see that all matrices $s_{m,n}$ are real whenever F is, and the same holds for the limits z_m. **QED**

Denote by $B'(\gamma)$ the ball in $(\mathbb{I} - \mathbb{E})\mathcal{A}_\gamma$ of radius $K\rho_0$, centered at the origin. Define $B(\gamma) = b_0 \oplus B'(\gamma)$, that is, $f \in \mathcal{A}_\gamma$ belongs to $B(\gamma)$ if and only if $\bar{f} \in b_0$ and $h = f - \bar{f}$ belongs to $B'(\gamma)$.

Consider now the set $\mathcal{M}_\gamma$ defined in Theorem 2.4, with $B(\gamma)$ as described above.

Corollary 5.5. *Let F be a family in the domain of $\widetilde{\mathfrak{N}}$, and let $s \in b_0$. Then $F(s)$ belongs to $\mathcal{M}_\gamma$ if and only if $s = z_0(F)$.*

Proof. Consider first $f = F(z_0)$. Set $f_n = F_n(z_n)$ for each $n > 0$. By the definition of $\mathfrak{N}_n$, and by Proposition 5.4, we have $f_n = \mathcal{N}_n(f_{n-1})$ for $n = 1, 2, \ldots$, and $\bar{f}_n = \mathbb{E}F_n(z_n) = z_n$ belongs to b_n. This shows that $f \in \mathcal{M}_\gamma$.

Consider now a fixed $s = s_0$ in b_0, and assume that $f_0 = F(s_0)$ belongs $\mathcal{M}_\gamma$. Then we can define $f_n = \widetilde{\mathfrak{N}}_n(f)$ for all $n > 0$, and $s_n = \bar{f}_n$ belongs to b_n. Set $F_0 = F$. Proceeding by induction, let $n > 0$, and assume $f_{n-1} = F_{n-1}(s_{n-1})$. Since $s_n = Y_n(s_{n-1})$, and since Y_n has a unique right inverse on b_n by Proposition 5.2, we have $s_{n-1} = Y_n^{-1}(s_n)$. As a result, $f_n = F_n(s_n)$. This shows that $s_n = Y_n(s_{n-1})$ holds for all $n > 0$, and thus $s_n = z_n$ by Proposition 5.4. **QED**

Proof of Theorem 2.4. To a function $h \in B'(\gamma)$ we associate the family $F : s \mapsto s + h$. This family belongs to the domain of $\widetilde{\mathfrak{N}}$. Now define $M(h) = z_0(F)$. By Corollary 5.5, $h + s = F(s)$ belongs to $\mathcal{M}_\gamma$ if and only if $s = M(h)$. This shows that $\mathcal{M}_\gamma$ is the graph of M over $B'(\gamma)$.

The analyticity of M follows from the analyticity of z_0. Furthermore, we have $M(0) = z_0(F^0) = 0$. The identity $DM(0) = 0$ follows from the fact that, by Proposition 5.2, the derivative of $F \mapsto Y_{n,F}^{-1}$ vanishes at F^0, for each $n \geq 0$. **QED**

The following estimate will be used in the next section. Denote by $\mathrm{I}_{n,m}$ the inclusion map from $\mathfrak{A}_m$ into $\mathfrak{A}_n$.

Proposition 5.6. *Let F be in the domain of $\widetilde{\mathfrak{N}}$. Then the map $Z'_n = Y_n \circ \cdots \circ Y_1$ satisfies*

$$\|DZ'_n(s) - \lambda_n^{-1}\mathrm{I}_{n,0}\| \leq 2^{-10}\|\lambda_n^{-1}\mathrm{I}_{n,0}\|\,, \tag{5.18}$$

for all s in the image of b_{n-1} under $Z_{0,n-1}$.

Proof. Define $s_{k-1} = Y_k^{-1}(s_k)$ for $k = n-1, \ldots 2, 1$, starting with a fixed but arbitrary $s_{n-1} \in b_{n-1}$. By using Proposition 5.2, and the fact that the inclusion map from $\mathfrak{A}_{k-1}$ into $\mathfrak{A}_k$ has norm $\rho_{k-1}/\rho_k = 4\eta_k \tau_k^{-\gamma_1}$, we obtain

$$\|DY_k(s_{k-1})\| \leq \bigl(1 + 2^{-4k-8}c_k\bigr)\|\eta_k^{-1}\mathrm{I}_{k,k-1}\|\,, \tag{5.19}$$

with $c_k = \tau_k^\gamma \|F_{k-1} - F^0\|_{k-1} < 1$. Taking products, the norm of $DZ'_k(s_0)$ can be bounded by twice the norm of $\lambda_k^{-1}\mathrm{I}_{k,0}$. Thus,

$$\begin{aligned}
\|DZ'_n(s_0) - \lambda_n^{-1}\mathrm{I}_{n,0}\| &= \|DY_n(s_{n-1}) \cdots DY_2(s_1)DY_1(s_0) - \lambda_n^{-1}\mathrm{I}_{n,0}\| \\
&\leq \sum_{k=1}^{n} \bigl\|\eta_n^{-1} \cdots \eta_{k+1}^{-1}\mathrm{I}_{n,k}\bigl[DY_k(s_{k-1}) - \eta_k^{-1}\mathrm{I}_{k,k-1}\bigr]DZ'_{k-1}(s_0)\bigr\| \\
&\leq \sum_{k=1}^{n} 2^{-4k-7}c_k \cdot \|\lambda_n^{-1}\mathrm{I}_{n,0}\|\,,
\end{aligned}$$

and the inequality (5.18) follows. **QED**

6. Reducibility

The main goal in this section is to prove Theorem 1.1.

Consider first the flow Φ_X for a general vector field $X = (\omega, f\,.)$ The identity

$$\Phi_X^t(q) = \mathrm{I} + \int_0^t f(q+s\omega)\Phi_X^s(q)\,ds\,. \tag{6.1}$$

can be used to construct and estimate Φ_X. By applying first the contraction mapping principle, and then the cocycle identity for Φ_X to improve the result, we obtain

$$\left\|\Phi_X^t - \mathrm{I}\right\|_\gamma \le e^{\|tf\|_\gamma} - 1\,. \tag{6.2}$$

This bound holds for any $\gamma \ge 0$, provided that $f \in \mathcal{A}_\gamma$.

Consider now $f_0 \in \mathcal{M}_\gamma$ and the corresponding renormalized functions $f_n = \widetilde{\mathcal{N}}_n(f_0)$. In order to simplify notation, the transformation U_{f_n} and the flow $\Phi_{(\omega_n, f_n)}$ will be denoted by U_n and Φ_n, respectively.

Lemma 6.1. *Let $f_0 \in \mathcal{M}_\gamma$. For each $n \ge 0$ there exists $V_n \in \mathcal{G}_0$ such that*

$$\Phi_n^t(q) = V_n(q+t\omega_n)^{-1}V_n(q)\,, \qquad t \in \mathbb{R}\,. \tag{6.3}$$

These function V_n satisfy the relations $V_{n+1} = (V_n \circ T_{n+1})U_n$ and the bounds

$$\|V_n - \mathrm{I}\|_0 \le 2^{4-n}\pi_n^{\gamma-\gamma_1}\sigma_1^{-1}\|f_0\|_\gamma\,. \tag{6.4}$$

Furthermore, the maps $f_0 \mapsto V_n$ are analytic.

Proof. By equation (2.6), we have

$$\Phi_n^t(q) = V_{m,n}(q+t\omega_n)^{-1}\Phi_m^{\eta_m\ldots\eta_{n+1}t}(T_m\ldots T_{n+1}q)V_{m,n}(q)\,, \tag{6.5}$$

for $m > n \ge 0$, where

$$V_{m,n}(q) = U_{m-1}(T_{m-1}\cdots T_{n+1}q)\cdots U_{n+1}(T_{n+1}q)U_n(q)\,. \tag{6.6}$$

For convenience later on, we also define $V_{n,n} = \mathrm{I}$. Using the notation of Section 5, we have $f_n \in 2b_n$ and thus

$$\|\eta_m\cdots\eta_{n+1}tf_n\|_\gamma \le 2\lambda_n^{-1}\lambda_m\rho_m|t| \le 2\cdot 4^{-m}\lambda_n^{-1}\rho_0|t|\,. \tag{6.7}$$

If m is sufficiently large, then (6.2) leads to the bound

$$\left\|\Phi_m^{\eta_m\ldots\eta_{n+1}t} - \mathrm{I}\right\|_0 \le 4^{1-m}\lambda_n^{-1}\rho_0|t|\,. \tag{6.8}$$

Thus, $\Phi_m^{\eta_m\ldots\eta_{n+1}t}$ converges in $\mathcal{G}_0$ to the identity, as $m \to \infty$, uniformly in t on compact subsets of $\mathbb{R}$.

Consider now the factors U_j in the product (6.6). By Theorem 2.2 and Theorem 2.3,

$$\sigma_{n+1}^{-1}\|h_n\|_\gamma \leq 2^{-n-9}\pi_n^{\gamma-\gamma_1}\varepsilon\,, \qquad \varepsilon = 2^9\sigma_1^{-1}\|f_0\|_\gamma < 1\,. \tag{6.9}$$

Combining this with the estimate (3.10), we obtain

$$\|U_n - \mathrm{I}\|_\gamma \leq 2^{-n-7}\pi_n^{\gamma-\gamma_1}\varepsilon\,, \qquad \|U_n\|_\gamma \leq e^{2^{-n-7}\varepsilon}\,. \tag{6.10}$$

Notice that $\|U\circ T\|_0 = \|U\|_0 \leq \|U\|_\gamma$ for any matrix T in $\mathrm{SL}(d,\mathbb{Z})$, and any U in $\mathcal{F}_\gamma$ with $\gamma \geq 0$. Thus, the bounds (6.10) can be used to estimate the product (6.6) in $\mathcal{G}_0$. We have

$$\|V_{m,n}\|_0 \leq \prod_{j=n}^{m-1}\|U_j\|_\gamma \leq e^{2^{-n-6}\varepsilon}\,, \tag{6.11}$$

and as a result,

$$\|V_{k,n} - V_{m,n}\|_0 \leq \sum_{j=m}^{k-1}\bigl\|(U_j - \mathrm{I})V_{j,n}\bigr\|_\gamma \leq 2^{-m-5}\pi_n^{\gamma-\gamma_1}\varepsilon\,, \tag{6.12}$$

for $k > m \geq n \geq 0$. This shows that the limits $V_n = \lim_{m\to\infty} V_{m,n}$ exist in $\mathcal{G}_0$, and that they have the properties described in Lemma 6.1. The analyticity of $f_0 \mapsto V_n$ follows from the uniform convergence of $V_{m,n} \to V_n$, combined with the fact that the map M defining the manifold $\mathcal{M}_\gamma$, the RG transformations $\mathcal{N}_n$, and the map $f \mapsto U_f$ described in Proposition 3.3, are all analytic. **QED**

Proposition 6.2. *The manifold $\mathcal{M}_\gamma$ is invariant under the torus-translations R_p, and the map $f_0 \mapsto V_0$ defined by Lemma 6.1 commutes with these translations.*

Proof. First, we note that the translations R_p are isometries on $\mathcal{F}$ and commute with $\mathbb{E}$. This shows in particular that $B(\gamma)$ is invariant under R_p.

The identity (3.15) shows that $R_p f_0$ belongs to the domain of $\widetilde{\mathcal{N}}_n$ whenever f_0 does, and that $\widetilde{\mathcal{N}}_n(f_0)$ and $\widetilde{\mathcal{N}}_n(R_p f_0)$ have the same torus-average. From the definition (2.14) of the sets $\widetilde{B}_n$ whose intersection is $\mathcal{M}_\gamma$, it is now clear that $\mathcal{M}_\gamma$ is invariant under torus-translations.

The fact that $f_0 \mapsto V_0$ commutes with R_p follows from an explicit computation, using the identities (3.15) and (6.6). **QED**

Lemma 6.3. *Let $\gamma \geq \gamma_2 > \gamma_1$ and $\varepsilon = \gamma - \gamma_2$. If $f_0 \in \mathcal{M}_\gamma$ then the function V_0 described in Lemma 6.1 belongs to $\mathcal{G}_\varepsilon$ and has a directional derivative $D_{\omega_0}V_0$ in $\mathcal{F}_\varepsilon$. As elements of $\mathcal{F}_\varepsilon$, both V_0 and $D_{\omega_0}V_0$ depend analytically on f_0. Furthermore, if f_0 is the restriction to $\mathbb{T}^d$ of an analytic function, then so is V_0.*

Proof. In order to avoid possible ambiguities, assume first that $\gamma = \gamma_2$. Denote by H and $\mathcal{H}$ the maps that associate to each $f \in B'(\gamma_2)$ via $f_0 = f + M(f)$ the corresponding function

V_0 and the value $V_0(0)$, respectively. Proposition 6.2 implies that $R_p V_0 = H(R_p f)$, and thus

$$V_0(p) = \mathcal{H}(R_p f)\,, \qquad p \in \mathbb{T}^d\,. \tag{6.13}$$

By Lemma 6.1 the function $\mathcal{H}$ is bounded and analytic on $B'(\gamma_2)$. Consider its Taylor series at zero,

$$\mathcal{H}(f) = \sum_{n=0}^{\infty} \mathcal{H}_n(f,\ldots,f)\,, \tag{6.14}$$

where $\mathcal{H}_n = D^n\mathcal{H}(0)/n!$. Let r be a fixed but arbitrary positive real number less than $K\rho_0$. Then the series (6.14) converges absolutely in the ball $\|f\|_{\gamma_2} \le r$, and the derivatives of $\mathcal{H}$ satisfy a bound $\|\mathcal{H}_n\| \le cr^{-n}$ as n-linear functionals on $\mathcal{A}_{\gamma_2}^n$.

Next, we allow $\gamma \ge \gamma_2$ but keep $\mathcal{H}$ as a function on $B'(\gamma_2)$. Concerning the condition $f_0 \in \mathcal{M}_\gamma$ in Lemma 6.3, we note that $\mathcal{M}_\gamma = \mathcal{M}_{\gamma_2} \cap B(\gamma)$, which follows from the definition (2.14) of the sets $\widetilde{B}_n$, and from the fact that $B(\gamma)$ is a subset of $B(\gamma_2)$.

Assume now that f belongs to $B'(\gamma)$ and satisfies $\|f\|_\gamma < r$. If we use the expansion

$$R_p f = \sum_{\nu\in\mathbb{Z}^d} F_\nu E_\nu(p)\,, \qquad F_\nu(q) = f_\nu E_\nu(q)\,, \quad E_\nu(q) = e^{i\nu\cdot q}\,, \tag{6.15}$$

where f_ν are the Fourier coefficients of f, then V_0 can be represented as follows:

$$V_0(p) = \sum_{n=0}^{\infty} \sum_{\nu_1,\ldots,\nu_n\in\mathbb{Z}^d} \mathcal{H}_n(F_{\nu_1},\ldots,F_{\nu_n}) E_{\nu_1}(p)\cdots E_{\nu_n}(p)\,. \tag{6.16}$$

By using the bound

$$\bigl|\mathcal{H}_n(F_{\nu_1},\ldots,F_{\nu_n})\bigr| \le cr^{-n} \prod_{j=1}^{n} \|F_{\nu_j}\|_{\gamma_2} = cr^{-n} \prod_{j=1}^{n} \|f_{\nu_j}\|\|E_{\nu_j}\|_{\gamma_2}\,, \tag{6.17}$$

and the fact that $\|E_\nu\|_{\gamma_2}\|E_\nu\|_\varepsilon = \|E_\nu\|_\gamma$, we obtain

$$\begin{aligned} \|V_0\|_\varepsilon &\le \sum_{n=0}^{\infty} \sum_{\nu_1,\ldots,\nu_n\in\mathbb{Z}^d} cr^{-n} \prod_{j=1}^{n} \|f_{\nu_j}\|\|E_{\nu_j}\|_\gamma \\ &= c\sum_{n=0}^{\infty}\biggl(r^{-1}\sum_{\nu\in\mathbb{Z}^d}\|f_\nu\|\|E_\nu\|_\gamma\biggr)^n = \frac{c}{1-r^{-1}\|f\|_\gamma}\,. \end{aligned} \tag{6.18}$$

This shows that $V_0 \in \mathcal{G}_\varepsilon$, as claimed. From the identities (6.1) and (6.3), we see that $D_{\omega_0}V_0$ belongs to $\mathcal{F}_\varepsilon$. The analytic dependence of V_0 (and thus $D_{\omega_0}V_0$) on f follows from the uniform convergence (6.18) of the Taylor expansion for $f \mapsto V_0$ on any ball $\|f\|_\gamma \le r'$ with $r' < r < K\rho_0$. Finally, if f is the restriction to $\mathbb{T}^d$ of an analytic function, then due to the exponential decay of the Fourier coefficients f_ν, we have

$$r^{-1}\sum_{\nu\in\mathbb{Z}^d}\|f_\nu\|\|E_\nu\|_{\gamma_2} e^{\delta\|\nu\|} < 1\,, \tag{6.19}$$

for $\delta > 0$ sufficiently small. By using this bound to estimate the sum (6.16), one finds that the sum is absolutely convergent in the region $|\mathrm{Im}(p_j)| < \delta/2$. Thus, V_0 extends analytically to this region. **QED**

The following lemma concerns the situation described in the introduction, where $f = f_0$ is of the form (1.10). These functions f define a closed linear subspace $\mathcal{A}^1_\gamma$ of $\mathcal{A}_\gamma$, which can also be characterized by the identity

$$f(q + (0,r)) = e^{-r\cdot J} f(q) e^{r\cdot J}\,, \qquad q \in \mathbb{T}^d\,, \quad r \in \mathbb{R}^\ell\,. \tag{6.20}$$

Lemma 6.4. *Let $\gamma \geq \gamma_2 + 1$ with $\gamma_2 > \gamma_1$, and assume that f_0 belongs to $\mathcal{M}_\gamma \cap \mathcal{A}^1_\gamma$. Set $f = f_0$ and $V = V_0$. If $g = \Theta_\star f$, then the flow for $Y = (w, g\,.)$ is given by equation (1.12), for some $C \in \mathfrak{A}$. The corresponding map $f_0 \mapsto C$ is analytic.*

Proof. The first equality in (1.12) follows from Lemma 6.1 and the definition (1.11). Define

$$\phi^t(x) = V(x + tw)\Phi^t_Y(x)V(x)^{-1}\,, \tag{6.21}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{T}^m$. Notice that ϕ is the flow for a skew system $Z = (w, h\,.)$ on $\mathbb{T}^m \times \mathfrak{G}$, and since $V \in \mathcal{G}_1$ by Lemma 6.3, the function h belongs to $\mathcal{A}_0$.

From the first equality in (1.12), we have

$$\phi^t(x) = V(x + tw)e^{tA}V(x + t\omega)^{-1}\,. \tag{6.22}$$

Consider now an arbitrary sequence $\{t_j\}$ such that $t_j\kappa \to 0$ on the torus $\mathbb{T}^\ell$, as $j \to \infty$. Then $\exp(t_j\kappa \cdot J) \to \mathrm{I}$. Furthermore, $\mathrm{dist}(t_j\omega, t_jw) \to 0$ on the torus $\mathbb{T}^d$, and since V is of class C^1, we have $\phi^{t+t_j}(x) \to \phi^t(x)$ uniformly in x, if $t = 0$. By the cocycle identity for the flow ϕ, the same holds for any $t \in \mathbb{R}$, and the convergence is uniform in t. This implies (see e.g. [22]) that the function $t \mapsto \phi^t(x)$ is periodic or quasiperiodic, with frequencies in $K = \{\kappa_1, \ldots, \kappa_\ell\}$. As a result,

$$h(x + tw) = \dot{\phi}^t(x)\phi^t(x)^{-1} \tag{6.23}$$

is also periodic or quasiperiodic in t, with frequencies in K. But the frequency module of $t \mapsto h(x + tw)$ is clearly a subset of $W = \{w_1, \ldots, w_m\}$, and since $W \cap K$ is empty, h has to be constant. Setting $C = h$, we obtain $\phi^t(x) = e^{tC}$, and the identity (1.12) now follows from (6.21). A computation of $h(x)$ from the equations (6.23) and (6.22) yields $C = VAV^{-1} - (D_\kappa V)V^{-1}$, evaluated at x. This identity (between matrices, if x is fixed), together with Lemma 6.3, shows that C depends analytically on f. **QED**

In order to complete the proof of Theorem 1.1, consider now the case where $\mathfrak{G}$ is a proper Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$. By Proposition 3.3, the restrictions to $\mathcal{A}$ of the transformations $\mathcal{N}_n$ take again values in $\mathcal{A}$, and so the transformations $\mathfrak{N}_n$ preserves the subspace of families taking values in $\mathcal{A}$. Thus, the map M described in Theorem 2.4 takes values in $\mathcal{A}$ when restricted to $\mathcal{A}$, as claimed in Theorem 1.1. Similarly, the fact that $U_f \in \mathcal{G}$ whenever $f = f_0 \in \mathcal{A}$ implies that the matrices (6.6) belong to $\mathfrak{G}$, and so the same is true for the limit $V = V_0(q)$. The same arguments apply to the case where $\mathfrak{G}$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, if we use that by Proposition 5.4, the parameter values z_0 defining the map M are all real in this case. The remaining claims of Theorem 1.1 now follow from Theorem 2.4, Lemma 6.1, and Lemma 6.3.

7. The special case $\mathfrak{G}$=SL(2,$\mathbb{R}$)

In this section, $\mathfrak{G}$ is the group of unimodular 2×2 matrices over $\mathbb{R}$, and $\mathfrak{A}$ is the corresponding Lie algebra of real traceless 2×2 matrices. As explained in the introduction, our approach to skew flows with nonzero fibered rotation number is to convert them to skew flows with zero (or near-zero) rotation number, which involves increasing the dimension of the torus. As far as renormalization is concerned, the main difficulty with this approach is that the space $\mathcal{A}^1$ of functions f of the form (1.10) is not invariant under renormalization. Superficially, the fact that the torus-average of $f\in\mathcal{A}^1$ is necessarily a constant multiple of J may seem to explain the statement about one-parameter families in Theorem 1.2. However, this property is neither invariant under renormalization, nor does is guarantee that the flow for $X=(\omega,f\,.)$ remains bounded. Below we will introduce an alternative property, that is more closely linked to hyperbolicity, and invariant.

First, we give a simple sufficient condition for a skew system to have nonzero fibered rotation number.

Proposition 7.1. *If $\det(f(q))>0$ for all $q\in\mathbb{T}^d$, then $\varrho(X)\neq 0$.*

Proof. If we set $\tau(t)=\operatorname{tr}(Jf(q))$ and $\delta(t)=\det(f(q))$, with $q=q_0+t\omega$, then an explicit calculation shows that (1.14) can be written as

$$2\dot{\alpha}=-\tau+\rho\sin(2\alpha+\beta)\,,\qquad \rho=\sqrt{\tau^2-4\delta}\,,\tag{7.1}$$

for some angle β depending on $f(q)$ and on u_0 . Notice that $\tau^2\geq 4\delta$, since $f(q)$ is traceless. Thus, if $f(q)$ is always elliptic ($\delta>0$), then $\dot{\alpha}$ is bounded away from zero and the rotation number cannot vanish. **QED**

One of our goals is to show that a vector field $X=(\omega,f\,.)$ with $f\in\mathcal{A}^1$ close to zero cannot generate a hyperbolic flow, by excluding the possibility that the renormalized functions f_n have the following property.

Definition 7.2. *Let S^1 be the set of unit vectors in $\mathbb{R}^2$. We say that a vector field $X=(\omega,f\,.)$ has the expanding cone property if for every $q\in\mathbb{T}^d$, there exists an open cone $\mathcal{C}(q)$ in $\mathbb{R}^2$ not intersecting its negative, with vertex at zero, and a unit vector $u(q)$ in this cone, such that the following holds. The map $q\mapsto S^1\cap\mathcal{C}(q)$ defines two continuous functions from $\mathbb{T}^d$ to S^1. The function $q\mapsto u(q)$ is continuous as well, and homotopic to a constant. Furthermore, for every $q\in\mathbb{T}^d$, the cone $\Phi_X^t(q)\mathcal{C}(q)$ is contained in $\mathcal{C}(q+t\omega)$ for all $t>0$, and the length of $\Phi_X^t(q)u(q)$ tends to infinity as $t\to\infty$.*

We note that the expanding cone property is invariant under coordinate changes of the form (2.1) or (1.5), with V continuous and homotopic to the identity. A simple condition that implies this property is given in the following proposition.

Proposition 7.3. *Assume that $f:\mathbb{T}^d\to\mathfrak{A}$ is continuous and of the form $f=C+h$, with $C\in\mathfrak{A}$ symmetric and $\|h(q)\|<\|C\|/4$ for all $q\in\mathbb{T}^d$. Then $X=(\omega,f\,.)$ has the expanding cone property.*

Proof. Our assumptions imply that the eigenvalues of C are $\pm\|C\|$. Let u_0 be a unit eigenvector of C for the eigenvalue $\|C\|$, and define $\mathcal{C}_0$ to be the set of all nonzero vectors in $\mathbb{R}^2$ whose angle with u_0 is less than $\pi/4$. Consider first the case $f \equiv C$. Then for every nonzero v on the boundary of $\mathcal{C}_0$, the vector fv points to the interior of the cone $\mathcal{C}_0$. Thus, the solutions of equation (1.13), with initial condition v_0 in $\mathcal{C}_0$, remains in $\mathcal{C}_0$ for all times $t > 0$. A straightforward computation shows that under the given assumptions of h, the same remains true for $f = C + h$. Thus, X has the expanding cone property, with the family of cones being $q \mapsto \mathcal{C}_0$, and with $u(q) = u_0$ for all q. Notice that no condition on ω is needed. **QED**

Lemma 7.4. *If f belongs to $\mathcal{A}^1$ then $X = (\omega, f\,.)$ cannot have the expanding cone property.*

Proof. Consider first an arbitrary $f \in \mathcal{A}$ such that $X = (\omega, f\,.)$ has the expanding cone property. Let $q \in \mathbb{T}^d$ be fixed. Using the notation of Definition 7.2, denote by $A(q)$ the set of all nonzero $v_0 \in \mathbb{R}^2$ such that $v(t) = \Phi_X^t(q)v_0$ belongs to $\mathcal{C}(q + tw)$ for some (and thus each sufficiently large) positive t. This set is clearly open. Notice that if v_0 is any nonzero vector in $\mathbb{R}^2$, with the property that $v(t) = \Phi_X^t(q)v_0$ tends to infinity as $t \to \infty$, then v_0 belongs to either $A(q)$ or $-A(q)$. This follows from the fact that $\Phi_X^t(q)$ is area-preserving (so the angle between $v(t)$ and $\Phi_X^t(q)u(q)$ has to approach zero), and that the opening angles of our cones are bounded away from zero. Thus, given that the two disjoint open sets $\pm A(q)$ cannot cover all of $\mathbb{R}^2 \setminus \{0\}$, it is not possible that $|v(t)| \to \infty$ as $t \to \infty$, for every nonzero $v_0 \in \mathbb{R}^2$.

Assume now for contradiction that f belongs to $\mathcal{A}^1$. Define $z_r(x) = e^{rJ}u(q)$, with u as described in Definition 7.2. Then $\Phi_Y^t(x)z_r(x) = e^{(r+t\kappa)J}\Phi_X^t(q)u(q)$ tends to infinity as $t \to \infty$. But as r increases from 0 to 2π, the vectors $z_r(x)$ cover all of S^1, since u is homotopic to a constant function. This implies that $\Phi_Y^t(x)v_0$ tends to infinity (in length) for each nonzero $v_0 \in \mathbb{R}^2$, which was shown above to be impossible. **QED**

Now we are ready to renormalize. Denote by $\mathfrak{J}$ the one-dimensional subspace of $\mathfrak{A}$, consisting of real multiples of the matrix J.

Lemma 7.5. *Let $h \in \mathcal{A}^1 \cap B'(\gamma)$, and define $F(s) = h + s$ for $s \in b_0$. Then the (unique) value $s = z_0(F)$ where the family F intersects $\mathcal{M}_\gamma$ belongs to $\mathfrak{J}$, and it is the unique matrix in $b_0 \cap \mathfrak{J}$ for which $F(s)$ has a zero fibered rotation number.*

Proof. Recall that $z_0 = z_0(F)$ is real, by Proposition 5.4. Assume for contradiction that z_0 does not belong to $\mathfrak{J}$. Then for sufficiently large m, the sets $Z_{0,m}(b_m/3)$ have an empty intersection with $\mathfrak{J}$. Denote by n the smallest value of m for which this intersection is empty, and define $\mathfrak{J}_k = Z_{0,n-1}(b_{n-1}/k) \cap \mathfrak{J}$.

Let $r = \|\eta_n^{-1}\mathrm{I}_{n,n-1}\|$. The bound (5.19) shows that the image under Y_n of $\frac{1}{3}b_{n-1}$ is contained in $\frac{r}{2}b_n$, and that the image of b_{n-1} contains $\frac{2r}{3}b_n$. The first property implies that $Z'_n(\mathfrak{J}_3)$ intersects $\frac{r}{2}b_n$ at some point s outside $\frac{1}{3}b_n$. Now consider the connected component of $Z'_n(\mathfrak{J}_1)$ containing s. By Proposition 5.6, this curve is sufficiently "parallel" to $\mathfrak{J}$ in order to intersect the subspace $\mathrm{tr}(J^*s) = 0$ at some point $s_n = Z'_n(s_0)$ that lies inside $\frac{2r}{3}b_n$, but outside $\frac{1}{4}b_n$. The matrix s_n is symmetric with norm $\geq \rho_n/4$, and by Lemma 5.3, we have $\|F_n(s_n) - s_n\| < \rho_n/16$. Thus, by Proposition 7.3, the vector field for $F_n(s_n)$ has the expanding cone property. Given that this property is invariant under coordinate changes of the form (2.1) or (1.5), with V continuous and homotopic to the identity, $F(s_0)$ has the same property. But since $s_0 \in \mathfrak{J}$, the function $F(s_0)$ belongs to $\mathcal{A}^1$, and we get a contradiction with Lemma 7.4. This shows that z_0 belongs to $\mathfrak{J}$.

Lemma 6.1 shows that $\varrho(F(z_0)) = 0$. Consider now $s_0 \in b_0 \cap \mathfrak{J}$ different from z_0. Then there exists $n > 0$ such that s_0 lies in $Z_{0,m}(b_m)$ for all $m < n$, but not in $Z_{0,n}(b_n)$, so the norm of $s_n = Z'_n(s_0)$ is at least ρ_n. On the other hand, $z_n = Z'_n(z_0)$ has norm less than $\rho_n/3$, as was shown in the proof of Proposition 5.4. Thus, $\|s_n - z_n\| > 2\|z_n\|$. Denote by B and C the symmetric and antisymmetric parts of $s_n - z_n$, respectively. By Proposition 5.6, we have $\|C\| > 10\|B\|$. In addition, $\|F_n(s_n) - s_n\| < \rho_n/16$ by Lemma 5.3. As a result, $\|C\| > \|F_n(s_n) - C\|$, which by Proposition 7.1 implies that $F_n(s_n)$ cannot have a vanishing fibered rotation number. Thus, we cannot have $\varrho(F(s_0)) = 0$, since this property is preserved under renormalization. **QED**

Proof of Theorem 1.2. We can follow the sketch given after the statement of this theorem. A straightforward computation shows that $\|L\|$ and $\|L^{-1}\|$ can be bounded by $2\|\kappa^{-1}A\|^{1/2}$ Thus, the indicated map $g \mapsto f = (\Theta_\star)^{-1}(cL^{-1}gL)$, with $c = \|\omega\|^{-1}$, admits the bound

$$\|f\| \le 2^{2\gamma}4c \cdot 4\|\kappa^{-1}A\|\|g - A\| \le 2^{2\gamma}16a\|g - A\|\,. \tag{7.2}$$

This shows that the image B_0 under $f \mapsto g$, of the domain $B = B(\gamma)$ for which Theorem 1.1 holds, contains a ball of radius $R = Kr_0/(2^{2\gamma}16a)$, centered at the constant function A. The remaining claims of Theorem 1.2 are now an immediate consequence of Theorem 1.1, Lemma 7.5, and Lemma 6.4. **QED**

References

[1] E.I. Dinaburg, Ja. G. Sinai, *The one-dimensional Schrödinger equation with quasiperiodic potential*, Funkcional. Anal. i Priložen. **9**, 8–21 (1975).

[2] J. Moser, J. Pöschel, *An extension of a result by Dinaburg and Sinaĭ on quasiperiodic potentials*, Comment. Math. Helv. **59**, 39–85 (1984).

[3] L.H. Eliasson, *Floquet solutions for the* 1*-dimensional quasi-periodic Schrödinger equation*, Commun. Math. Phys. **146**, 447–482 (1992).

[4] R. Krikorian, *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts*, Astérisque **259**, vi+216 (1999).

[5] R. Krikorian, *Réductibilité presque partout des flots fibrés quasi-périodiques à valeurs dans des groupes compacts*, Ann. Sci. École Norm. Sup. (4) **32**, 187–240 (1999).

[6] R. Krikorian, *C^0-densité globale des systèmes produits-croisés sur le cercle réductibles*, Erg. Theor. Dyn. Syst. **19**, 61–100 (1999).

[7] R. Krikorian, *Global density of reducible quasi-periodic cocycles on* $\mathbf{T}^1 \times \mathrm{SU}(2)$, Ann. of Math. (2) **154**, 269–326 (2001).

[8] M. Rychlik, *Renormalization of cocycles and linear ODE with almost-periodic coefficients*, Invent. Math. **110**, 173–206 (1992).

[9] L.H. Eliasson, *Ergodic skew-systems on* $\mathbb{T}^d \times \mathrm{SO}(3, \mathbb{R})$, Erg. Theor. Dyn. Syst. **22**, 1429–1449 (2002).

[10] A. Ávila, R. Krikorian, *Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles*, Ann. of Math., to be published (2005).

[11] J. Lopes Dias, *Renormalization and reducibility of Brjuno skew-systems*, preprint (2004).

[12] H. Koch, *A renormalization group for Hamiltonians, with applications to KAM tori*, Erg. Theor. Dyn. Syst. **19**, 1–47 (1999).

[13] J. Lopes Dias, *Renormalisation of flows on the multidimensional torus close to a KT frequency vector*, Nonlinearity **15**, 647–664, (2002).

[14] J. Lopes Dias, *Renormalisation scheme for vector fields on T^2 with a Diophantine frequency*, Nonlinearity **15**, 665–679, (2002).

[15] C. Chandre and H.R. Jauslin, *Renormalization–group analysis for the transition to chaos in Hamiltonian systems*, Physics Reports **365**, 1–64, (2002).

[16] H. Koch, *A renormalization group fixed point associated with the breakup of golden invariant tori*, Discrete Contin. Dynam. Systems A **8**, 633–646 (2002).

[17] D.G. Gaidashev, *Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori*, Discrete Contin. Dynam. Systems A **13**, 63–102 (2005).

[18] S. Kocić, *Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori*, Nonlinearity **18**, 2513–2544 (2005).

[19] K. Khanin, J. Lopes Dias, J. Marklof, *Multidimensional continued fractions, dynamical renormalization and KAM theory*, preprint (2005).

[20] J.W.S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press (1957).

[21] E. Hille, R.S. Phillips, *Functional analysis and semi–groups*, AMS Colloquium Publications, **31** (1974).

[22] A. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, **377**, Springer Verlag, Berlin · Heidelberg · New York (1974).

[23] R. Johnson, J. Moser, *The rotation number for almost periodic potentials*, Commun. Math. Phys. **84**, 403–438 (1982).

[24] J.C. Lagarias, *Geodesic multidimensional continued fractions*, Proc. London Math. Soc. **69**, 464–488 (1994).

[25] D.Y. Kleinbock, G.A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. of Math. (2) **148**, 339–360 (1998).