BRJUNO CONDITION AND RENORMALIZATION FOR POINCARÉ FLOWS

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Abstract. In this paper we give a new proof of the local analytic linearisation of flows on $\mathbb{T}^2$ with a Brjuno rotation number, using renormalization techniques.

1. Introduction

We define a renormalization scheme for analytic vector fields on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The vector fields are required to generate flows of Poincaré type, i.e. there is a classification by a unique asymptotic slope α (winding ratio) of their lifts to the universal cover. This winding ratio α is invariant under coordinate transformations of the torus, up to the action of $\mathrm{GL}(2,\mathbb{Z})$, and the renormalization acts on it as the Gauss map. So, all vector fields with $\alpha \neq 0$ are renormalisable, and those with $\alpha \in \mathbb{R} - \mathbb{Q}$ are infinitely renormalisable.

The renormalization of flows methodology has been used in several contexts such as Hamiltonian systems e.g. [9, 16, 4, 3, 10, 11, 6, 13, 12], and toroidal flows [14, 15]. In all these works the frequency vectors of the quasiperiodic motions considered are Diophantine, and in some cases subsets of these of constant, Koch and golden types. The present approach includes an improved version of the renormalization operators in [14, 15], done to extend the result in [15] to Brjuno winding ratios. This can then be applied to other quasi-periodic problems such as the ones mentioned above, extending them to Brjuno frequency vectors in the lower dimensional case.

The problem we consider in this paper is the analytic "rectification" of the flow generated by a close to constant vector field. That is to find an analytic conjugacy between a given flow and a linear one with the same winding ratio. This is equivalent to the conjugacy to pure rotation problem in the context of circle diffeomorphisms, for which the rotation number takes the role of the winding ratio. (The two systems are related by considering the return map to a transversal of the flow.) Yoccoz [17] found that the set of Brjuno numbers is exactly the set of frequencies for which one can guarantee such linearisation in the local analytic case. Even if this was done for the Siegel problem on linearisation of holomorphic maps in the neighbourhood of a fixed point, the same holds in the circle map ([5]) and Poincaré flow contexts. In this paper we recover the sufficient part of Yoccoz's result corresponding to [1, 2].

We say that two flows ϕ_t and ψ_t on $\mathbb{T}^2$ are C^r-conjugate (or orbit equivalent) if there is a C^r-diffeomorphism h of $\mathbb{T}^2$ taking orbits of ϕ_t onto those of ψ_t, preserving orientation. Notice that we allow a time change τ giving more satisfying conjugacy classes. This is the same to say that two vector fields X, Y on $\mathbb{T}^2$ are conjugate if $(\frac{d}{dt}\tau)\, X \circ h = Dh\, Y$.

Theorem 1.1. *Take the flow generated by a real-analytic vector field $\boldsymbol{v}$ on $\mathbb{T}^2$ sufficiently close to the constant vector field $\boldsymbol{\omega} \in \mathbb{R}^2$ and with the same winding ratio α. Then it is*

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analytically conjugate to the linear flow $\phi_t\colon \boldsymbol{x} \mapsto \boldsymbol{x} + t\boldsymbol{\omega}$ *on* $\mathbb{T}^2$, $t \geq 0$, *if* α *is a Brjuno number. Moreover, the conjugacy depends analytically on* $\boldsymbol{v}$.

We show the above theorem by relating it to the fact that the orbit under renormalization of a constant vector field with Brjuno winding ratio attracts all the nearby orbits in the same homotopy class. Recall that Brjuno numbers include all Diophantines and some Liouville, and is a full Lebesgue measure set. Finally, we remark that the convergence of the renormalization does not require the use of its derivatives as in [9, 14, 15]. Thus, it should be useful to the study of C^r vector fields for which the renormalization is not C^1.

In section 2 we review the basic properties of the continued fraction expansion of irrational numbers. We construct the renormalization scheme and proof its convergence in section 3. The construction of the conjugacy is done in section 4. Finally, in the appendix A we present a proof of Theorem 3.4 on the elimination of non-resonant modes, using a homotopy method.

2. Continued fractions

2.1. **Gauss map.** Consider an irrational number $0 < \alpha = \alpha_0 < 1$ written in its continued fractions expansion:

$$\alpha = [a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \tag{2.1}$$

$a_n \in \mathbb{N}$. Its iterates under the Gauss map are $\alpha_n = \{\alpha_{n-1}^{-1}\} = [a_{n+1}, \dots]$, $n \in \mathbb{N}$, that is

$$\alpha_n = \frac{1}{a_{n+1} + \alpha_{n+1}}. \tag{2.2}$$

Let $\beta_n = \prod_{i=0}^n \alpha_i$, $n \in \mathbb{N} \cup \{0\}$. It is a well-known fact (cf. [7]) that

$$\beta_n \leq \gamma^n, \quad \gamma = (\sqrt{5} - 1)/2. \tag{2.3}$$

Consider the transfer matrices in $\mathrm{GL}(2, \mathbb{Z})$:

$$T^{(n)} = \begin{bmatrix} -a_n & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.4}$$

In addition, define $P^{(0)} = I$ and

$$P^{(n)} = T^{(n)} \dots T^{(1)} = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}^{-1}, \quad n \in \mathbb{N}. \tag{2.5}$$

This gives the rational approximants $p_n/q_n = [a_1, \dots, a_n] \in \mathbb{Q}$ with

$$\frac{1}{2q_{n+1}} \leq \beta_n \leq \frac{1}{q_{n+1}}, \tag{2.6}$$

with a similar relation for p_n.

Finally, define the sequences of vectors in $\mathbb{R}^2$:

$$\begin{aligned} \boldsymbol{\omega}^{(n)} &= \alpha_{n-1}^{-1} T^{(n)} \boldsymbol{\omega}^{(n-1)} = (\alpha_n, 1) \\ \boldsymbol{\Omega}^{(n)} &= -\alpha_{n-1}^{-1}\, {}^\top T^{(n)^{-1}} \boldsymbol{\Omega}^{(n-1)} = (1, -\alpha_n). \end{aligned} \tag{2.7}$$

2.2. **Brjuno condition.** An irrational number α is a Brjuno number if

$$\sum_{n\geq 1} \frac{\log(q_{n+1})}{q_n} < +\infty. \tag{2.8}$$

The set of all Brjuno numbers is denoted by BC.

2.3. **Hyperbolicity of the transfer matrices.** As we shall see, a crucial step in our renormalization scheme is to eliminate all far from resonance modes in the Fourier series, i.e., all terms labeled by integer vectors outside the cone

$$I_n^+ = \{\boldsymbol{k} \in \mathbb{Z}^2 : |\boldsymbol{k}\cdot\boldsymbol{\omega}^{(n)}| \leq \sigma_n \|\boldsymbol{k}\|\} \tag{2.9}$$

for a given $\sigma_n > 0$, $n \in \mathbb{N}\cup\{0\}$. We use the norm in $\mathbb{R}^2$ given by $\|(x_1,x_2)\| = |x_1| + |x_2|$ and the matrix norm $\|A\| = \max_j \sum_i |A_{i,j}|$ for a square matrix $A = [A_{i,j}]$.

Lemma 2.1. *For all $\boldsymbol{k} \in I_{n-1}^+$ and $n \in \mathbb{N}$, we have*

$$\| {}^{\top}{T^{(n)}}^{-1}\boldsymbol{k}\| \leq A_{n-1}\|\boldsymbol{k}\|, \tag{2.10}$$

where

$$A_{n-1} = \sigma_{n-1}\|{T^{(n)}}^{-1}\| + \alpha_{n-1}\frac{\|\boldsymbol{\Omega}^{(n)}\|}{\|\boldsymbol{\Omega}^{(n-1)}\|}. \tag{2.11}$$

Proof. We write $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, where

$$\boldsymbol{k}_1 = \frac{\boldsymbol{k}\cdot\boldsymbol{\omega}^{(n-1)}}{\boldsymbol{\omega}^{(n-1)}\cdot\boldsymbol{\omega}^{(n-1)}}\,\boldsymbol{\omega}^{(n-1)}, \qquad \boldsymbol{k}_2 \in \operatorname{span}\{\boldsymbol{\Omega}^{(n-1)}\}. \tag{2.12}$$

Firstly,

$$\| {}^{\top}{T^{(n)}}^{-1}\boldsymbol{k}_1\| = \frac{\| {}^{\top}{T^{(n)}}^{-1}\boldsymbol{\omega}^{(n-1)}\|}{|\boldsymbol{\omega}^{(n-1)}\cdot\boldsymbol{\omega}^{(n-1)}|}|\boldsymbol{k}\cdot\boldsymbol{\omega}^{(n-1)}| \leq \sigma_{n-1}\|{T^{(n)}}^{-1}\|\,\|\boldsymbol{k}\| \tag{2.13}$$

since $\boldsymbol{k} \in I_{n-1}^+$ and

$$\frac{\| {}^{\top}{T^{(n)}}^{-1}\boldsymbol{\omega}^{(n-1)}\|}{|\boldsymbol{\omega}^{(n-1)}\cdot\boldsymbol{\omega}^{(n-1)}|} = \frac{1+\alpha_{n-1}+a_n}{(1+\alpha_{n-1}^2)\|{T^{(n)}}^{-1}\|}\|{T^{(n)}}^{-1}\| \leq \|{T^{(n)}}^{-1}\|. \tag{2.14}$$

Secondly, using

$$\| {}^{\top}{T^{(n)}}^{-1}\boldsymbol{k}_2\| = \alpha_{n-1}\frac{\|\boldsymbol{\Omega}^{(n)}\|}{\|\boldsymbol{\Omega}^{(n-1)}\|}\|\boldsymbol{k}_2\|, \tag{2.15}$$

we get (2.10). □

3. Renormalization of vector fields

3.1. **Definitions.** The transformation of a vector field X on a manifold M by a diffeomorphism $\psi\colon M \to M$ is given by the so-called *pull-back* of X under ψ:

$$\psi^* X = (D\psi)^{-1} X \circ \psi.$$

As the tangent bundle of the 2-torus is trivial, $T\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$, we identify the set of vector fields on $\mathbb{T}^2$ with the set of functions from $\mathbb{T}^2$ to $\mathbb{R}^2$, that can be regarded as maps of $\mathbb{R}^2$ by lifting to the universal cover. We will make use of the analyticity to extend to the complex domain, so we will deal with complex analytic functions.

In the following, $A \ll B$ means 'there is a constant $C > 0$ such that $A \leq CB$'.

3.2. **Spaces of vector fields.** Let $\rho > 0$ and consider the domain

$$\mathcal{D}_\rho = \{\boldsymbol{x} \in \mathbb{C}^2 \colon \|\operatorname{Im}\boldsymbol{x}\| < \rho/2\pi\}, \tag{3.1}$$

for the norm $\|\boldsymbol{u}\| = \sum_i |u_i|$ on $\mathbb{C}^2$. Take a $\mathbb{Z}^2$-periodic complex analytic function $f\colon \mathcal{D}_\rho \to \mathbb{C}^2$ on the form of the Fourier series

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} f_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

with $f_{\boldsymbol{k}} \in \mathbb{C}^2$. The Banach spaces $\mathcal{A}_\rho$ and $\mathcal{A}'_\rho$ are the subspaces such that the respective norms

$$\|f\|_\rho = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \|f_{\boldsymbol{k}}\| \,\mathrm{e}^{\rho\|\boldsymbol{k}\|}, \tag{3.2}$$

$$\|f\|'_\rho = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} (1 + 2\pi\|\boldsymbol{k}\|)\,\|f_{\boldsymbol{k}}\|\,\mathrm{e}^{\rho\|\boldsymbol{k}\|} \tag{3.3}$$

are finite. A similar Banach space is composed by $\mathbb{C}$-valued functions related to the ones in $\mathcal{A}_\rho$. The norm $|\cdot|_\rho$ on this space is related to $\|\cdot\|_\rho$ in the obvious way such that $\|f\|_\rho = \sum_{i=1}^2 |f_i|_\rho$, where $f = (f_1, f_2)$.

Some of the properties of the above spaces are of easy verification. For any $f, g \in \mathcal{A}'_\rho$:

- $|f \cdot g|_\rho \le \|f\|_\rho \,\|g\|_\rho$,
- $\|f(\boldsymbol{x})\| \le \|f\|_\rho \le \|f\|'_\rho$ where $\boldsymbol{x} \in \mathcal{D}_\rho$,
- $\|f\|_{\rho-\delta} \le \|f\|_\rho$ with $\delta < \rho$.

Let $\boldsymbol{\omega} \in \mathbb{R}^2 - \{0\}$. In the following, we will be studying vector fields of the form

$$X(\boldsymbol{x}) = \boldsymbol{\omega} + f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathcal{D}_\rho, \tag{3.4}$$

where $f \in \mathcal{A}_\rho$.

3.3. **Notion of analyticity.** We will be using maps between Banach spaces over $\mathbb{C}$ with a notion of analyticity stated as follows (cf. e.g. [8]): a map F defined on a domain is analytic if it is locally bounded and Gâteaux differentiable. If it is analytic on a domain, it is continuous and Fréchet differentiable. Moreover, we have a convergence theorem which is going to be used later on. Let $\{F_k\}$ be a sequence of functions analytic and uniformly locally bounded on a domain D. If $\lim_{k\to+\infty} F_k = F$ on D, then F is analytic on D.

3.4. **Spatial and time rescalings.** The fundamental step of the renormalization is a rescaling of the domain of definition of our vector fields. This is done by a linear transformation coming essentially from the continued fraction expansion of $\boldsymbol{\omega} = \boldsymbol{\omega}^{(0)}$. In addition, we perform a linear reparametrisation of time because the orbits take longer to cross the new torus.

Let $\rho_{n-1} > 0$ and fix an arbitrary vector field of the form

$$X(\boldsymbol{x}) = \boldsymbol{\omega}^{(n-1)} + f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathcal{D}_{\rho_{n-1}}, \tag{3.5}$$

with $f \in \mathcal{A}_{\rho_{n-1}}$, $n \in \mathbb{N}$. Write the constant Fourier terms through the projection

$$\mathbb{E}f = \int_{\mathbb{T}^2} f(\boldsymbol{x})d\boldsymbol{x} = f_0. \tag{3.6}$$

We are interested in the following coordinate and time changes:

$$L_n \colon \boldsymbol{x} \mapsto {T^{(n)}}^{-1}\boldsymbol{x}, \qquad t \mapsto \tau_n(f_0)t, \tag{3.7}$$

where

$$\tau_n(f_0)=\frac{\boldsymbol{\omega}^{(n)}\cdot\boldsymbol{\omega}^{(n)}}{\boldsymbol{\omega}^{(n)}\cdot\mathbb{E}L_n^*X}\quad\text{and}\quad \mathbb{E}L_n^*X=\alpha_{n-1}\boldsymbol{\omega}^{(n)}+T^{(n)}f_0. \tag{3.8}$$

The vector field in the new coordinates is the image of the map

$$X\mapsto\mathcal{L}_n(X)=\tau_n(f_0)\,L_n^*X.$$

That is, for $\boldsymbol{x}\in L_n^{-1}\mathcal{D}_{\rho_{n-1}}$,

$$\mathcal{L}_n(X)=\boldsymbol{\omega}^{(n)}+\widehat{\mathcal{L}}_n(f_0)+\widetilde{\mathcal{L}}_n(X) \tag{3.9}$$

where

$$\begin{aligned}\widehat{\mathcal{L}}_n(f_0)&=\tau_n(f_0)(\mathbb{I}-\mathbb{P}_n)L_n^*f_0,\\ \widetilde{\mathcal{L}}_n(X)&=\tau_n(f_0)L_n^*(f-f_0),\end{aligned} \tag{3.10}$$

where $\mathbb{P}_n$ stands for the projection of $\mathcal{A}_{\rho_n}$ over $\boldsymbol{\omega}^{(n)}$.

Notice that $\mathcal{L}_n(c\,\boldsymbol{\omega}^{(n-1)})=\boldsymbol{\omega}^{(n)}$, for any $c\neq 0$. Moreover, if the winding ratio X is α_{n-1}^{-1}, then the winding ratio of $\mathcal{L}_n(X)$ is α_n^{-1}.

3.5. **Far from resonance modes.** Given $\sigma_n>0$, we define the *far from resonance modes* with respect to $\boldsymbol{\omega}^{(n)}$ to be the ones whose indices are in the cone $I_n^-=\mathbb{Z}^2-I_n^+$. It is also useful to define the projections $\mathbb{I}_n^+$ and $\mathbb{I}_n^-$ by restricting the Fourier modes to I_n^+ and I_n^-, respectively. The identity operator is $\mathbb{I}=\mathbb{I}_n^++\mathbb{I}_n^-$.

3.6. **Improvement of analyticity.** We now restrict to the set of $f\in\mathcal{A}_{\rho_{n-1}}$ such that

$$\|f_0\|\leq\frac{\alpha_{n-1}\beta_{n-1}|\boldsymbol{\omega}^{(n)}\cdot\boldsymbol{\omega}^{(n)}|}{(1+\beta_{n-1})\|\,{}^\top T^{(n)}\boldsymbol{\omega}^{(n)}\|}. \tag{3.11}$$

So, we can estimate $\tau_n(f_0)$ by

$$|\tau_n(f_0)|\leq\left[\alpha_{n-1}-\left|\frac{\boldsymbol{\omega}^{(n)}\cdot T^{(n)}f_0}{\boldsymbol{\omega}^{(n)}\cdot\boldsymbol{\omega}^{(n)}}\right|\right]^{-1}\leq\frac{1+\beta_{n-1}}{\alpha_{n-1}}. \tag{3.12}$$

Proposition 3.1. *If $\delta>0$ and*

$$0<\rho_n'\leq\frac{\rho_{n-1}}{A_{n-1}}-\delta, \tag{3.13}$$

then $\widetilde{\mathcal{L}}_n$ as a map from $\mathbb{I}_{n-1}^+\mathcal{A}_{\rho_{n-1}}$ into $(\mathbb{I}-\mathbb{E})\mathcal{A}'_{\rho_n'}$ is continuous with

$$\|\widetilde{\mathcal{L}}_n(X)\|\leq 2\left(1+\frac{2\pi}{\mathrm{e}\delta}\right)\frac{\|T^{(n)}\|}{\alpha_{n-1}}\|(\mathbb{I}-\mathbb{E})X\|_{\rho_{n-1}}, \tag{3.14}$$

for every $X\in\mathbb{I}_{n-1}^+\mathcal{A}_{\rho_{n-1}}$

Proof. With $X=\boldsymbol{\omega}^{(n-1)}+f\in\mathbb{I}_{n-1}^+\mathcal{A}_{\rho_{n-1}}$, the claim follows from

$$\begin{aligned}\|\widetilde{\mathcal{L}}_n(X)\|'_{\rho_n'}&\leq|\tau_n(f_0)|\,\|T^{(n)}\|\sum_{\boldsymbol{k}\in I_{n-1}^+-\{0\}}\left(1+2\pi\|\,{}^\top {T^{(n)}}^{-1}\boldsymbol{k}\|\right)\|f_{\boldsymbol{k}}\|\mathrm{e}^{\rho_n'\|\,{}^\top {T^{(n)}}^{-1}\boldsymbol{k}\|}\\ &\leq 2(1+2\pi\mathrm{e}^{-1}\delta^{-1})\alpha_{n-1}^{-1}\|T^{(n)}\|\sum_{\boldsymbol{k}\in I_{n-1}^+-\{0\}}\|f_{\boldsymbol{k}}\|\exp\left[(\rho_n'+\delta)\|\,{}^\top {T^{(n)}}^{-1}\boldsymbol{k}\|\right]\\ &\leq 2(1+2\pi\mathrm{e}^{-1}\delta^{-1})\alpha_{n-1}^{-1}\|T^{(n)}\|\sum_{\boldsymbol{k}\in I_{n-1}^+-\{0\}}\|f_{\boldsymbol{k}}\|\exp\left[(\rho_n'+\delta)A_{n-1}\|\boldsymbol{k}\|\right]\\ &\leq 2(1+2\pi\mathrm{e}^{-1}\delta^{-1})\alpha_{n-1}^{-1}\|T^{(n)}\|\,\|(\mathbb{I}-\mathbb{E})f\|_{\rho_{n-1}},\end{aligned} \tag{3.15}$$

where we have used Lemma 2.1 and the inequality $t \le (\mathrm{e}\delta)^{-1}\mathrm{e}^{\delta t}$, $t \ge 0$. □

3.7. Estimate on the constant modes.

Proposition 3.2. *Suppose that* $X = \boldsymbol{\omega}^{(n-1)} + f \in \mathcal{A}_{\rho_{n-1}}$ *with winding ratio* α_{n-1}^{-1}. *Then*

$$\|\widehat{\mathcal{L}}_n(f_0)\| \le \|\widetilde{\mathcal{L}}_n(X)\|_{\rho_n'}.$$

Proof. We are going to show that, under the above hypothesis, $Y = \mathcal{L}_n(X) \in \mathcal{A}_{\rho_n'}$ with winding ratio α_n^{-1} belongs to the subset

$$C_n = \left\{ Z \in \mathcal{A}_{\rho_n'} \colon \|(\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(Z)\| \le \|(\mathbb{I} - \mathbb{E})Z\|_{\rho_n'} \right\}.$$

A set of vector fields D_n that do not cross the line spanned by $\boldsymbol{\omega}^{(n)}$ can be of the form:

$$D_n = \left\{ Z \in \mathcal{A}_{\rho_n'} \colon \|Z(\boldsymbol{x}) - \mathbb{E}\,Z\| < \|(\mathbb{I} - \mathbb{P}_n) \circ \mathbb{E}\,(Z)\|, \boldsymbol{x} \in \mathcal{D}_{\rho_n'} \right\}.$$

The slopes of all the vectors $Y(\boldsymbol{x})$ are bigger than α_n^{-1} or always less than α_n^{-1}, never crossing that value (as for their respective winding ratio). Since $\|Y(\boldsymbol{x}) - \mathbb{E}\,Y\| \le \|(\mathbb{I} - \mathbb{E})Y\|_{\rho_n'}$ for every $\boldsymbol{x} \in \mathcal{D}_{\rho_n'}$, the complementary set of D_n, contained in C_n, includes all (but not only) vector fields with the same winding ratio as $\boldsymbol{\omega}^{(n)}$. □

3.8. Cut-off of the analyticity strip.

Let $0 < \rho_n'' < \rho_n'$. Consider the inclusion operator $\mathcal{I}_n \colon \mathcal{A}_{\rho_n'}' \to \mathcal{A}_{\rho_n''}'$ by restricting $X \in \mathcal{A}_{\rho_n'}'$ to a smaller domain $X|\mathcal{D}_{\rho_n''} \in \mathcal{A}_{\rho_n''}'$.

Proposition 3.3. *If*

$$0 < C < \|T^{(n)}\|/(\sigma_n^2 \alpha_{n-1}) \quad \textit{and} \quad 0 < \rho_n'' \le \rho_n' - \log(\|T^{(n)}\|/C\sigma_n^2\alpha_{n-1}),$$

then

$$\|\mathcal{I}_n(\mathbb{I} - \mathbb{E})\| \le C\sigma_n^2\alpha_{n-1}\|T^{(n)}\|^{-1}.$$

Proof. The assertion follows immediately. □

3.9. Elimination of far from resonance modes.

The theorem below (to be proven in Section A.1) states the existence of a nonlinear change of coordinates isotopic to the identity, that gives a new vector field without far from resonance I_n^- modes. That is, the vector field contains only resonant modes. The fact that we are restricting the eliminattion to the far from resonance modes avoids dealing with the usual problems related to small divisors.

For given $\rho_n, \varepsilon, \nu > 0$, denote by $\mathcal{V}_\varepsilon$ the open ball in $\mathcal{A}_{\rho_n+\nu}'$ centred at $\boldsymbol{\omega}^{(n)}$ with radius ε.

Theorem 3.4. *Let* $\sigma_n < \|\boldsymbol{\omega}^{(n)}\|$ *and*

$$\varepsilon_n = \frac{\sigma_n}{42} \min\left\{ \frac{\nu}{4\pi}, \frac{\sigma_n}{72\|\boldsymbol{\omega}^{(n)}\|} \right\}. \tag{3.16}$$

For all $X \in \mathcal{V}_{\varepsilon_n}$ *there exists an isotopy* $U_t = \mathrm{Id} + u_t \colon \mathcal{D}_{\rho_n} \to \mathcal{D}_{\rho_n+\nu}$, $t \in [0,1]$, *of analytic diffeomorphisms in* $\mathcal{A}_{\rho_n}'$ *satisfying*

$$\mathbb{I}_n^- U_t^*(X) = (1-t)\,\mathbb{I}_n^- X, \qquad U_0 = \mathrm{Id}\,. \tag{3.17}$$

This defines the maps

$$\begin{aligned} \mathfrak{U}_t \colon \mathcal{V}_{\varepsilon_n} &\to \mathcal{A}_{\rho_n}' \\ X &\mapsto U_t \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \mathcal{U}_t\colon \mathcal{V}_{\varepsilon_n} &\to \mathbb{I}_n^+\mathcal{A}_{\rho_n} \oplus (1-t)\mathbb{I}_n^-\mathcal{A}'_{\rho_n+\nu} \\ X &\mapsto U_t^*(X) \end{aligned} \tag{3.19}$$

which are analytic, and satisfy the inequalities

$$\begin{aligned} \|\mathfrak{U}_t(X) - \operatorname{Id}\|'_{\rho_n} \leq& \frac{42t}{\sigma_n}\|\mathbb{I}_n^- X\|_{\rho_n} \\ \|\mathcal{U}_t(X) - \boldsymbol{\omega}^{(n)}\|_{\rho_n} \leq& (3-t)\|X - \boldsymbol{\omega}^{(n)}\|'_{\rho_n+\nu}. \end{aligned} \tag{3.20}$$

If X is real-analytic, then $\mathfrak{U}_t(X)(\mathbb{R}^2) \subset \mathbb{R}^2$.

3.10. **Renormalization scheme.** A convenient choice for the width of the resonance cones I_n^+ is

$$\sigma_n = \frac{\alpha_n\beta_n\|\boldsymbol{\Omega}^{(n+1)}\|}{\|T^{(n+1)^{-1}}\|\,\|\boldsymbol{\Omega}^{(n)}\|}, \qquad n \in \mathbb{N}\cup\{0\}. \tag{3.21}$$

Define B_n by the product

$$B_n = A_0 \dots A_n, \tag{3.22}$$

where $A_k \leq \alpha_k(1+\beta_k)\|\boldsymbol{\Omega}^{(k+1)}\|\,\|\boldsymbol{\Omega}^{(k)}\|^{-1}$ as in Lemma 2.1. Hence, we have constants $c_1, c_2 > 0$ (independent of n) yielding

$$c_1 \leq \frac{B_n}{\beta_n} \leq c_1\prod_{i=0}^{n}(1+\beta_i) \leq c_2. \tag{3.23}$$

Therefore, using (2.6) and (2.8), $\alpha \in BC$ iff

$$\sum_{i=1}^{+\infty} B_{i-1}\log\left(\frac{\|T^{(i)}\|}{\sigma_i^2\alpha_{i-1}}\right) < +\infty. \tag{3.24}$$

Fix δ and ν as in Proposition 3.1 and Theorem 3.4, respectively. For $\rho_0 > 0$ and some sufficiently small constant $C > 0$ (to be chosen later and depending only on δ and ν), take the sequence

$$\rho_n = \frac{1}{B_{n-1}}\left[\rho_0 - \sum_{i=1}^{n} B_{i-1}\log\left(\frac{\|T^{(i)}\|}{C\sigma_i^2\alpha_{i-1}}\right) - \sum_{i=1}^{n} B_{i-1}(\nu+\delta)\right]. \tag{3.25}$$

Define now the function

$$\mathcal{B}(\alpha) = \sum_{i=1}^{+\infty} B_{i-1}\log\left(\frac{\|T^{(i)}\|\mathrm{e}^{\nu+\delta}}{C\sigma_i^2\alpha_{i-1}}\right). \tag{3.26}$$

This means that $\alpha \in BC$ iff $\mathcal{B}(\alpha) < +\infty$. So, if $\rho_0 > \mathcal{B}(\alpha)$, there exists $R > 0$ such that

$$\rho_n \geq \frac{R}{\beta_{n-1}}. \tag{3.27}$$

The *one-step renormalization* operator is

$$\mathcal{N}_n = \mathcal{U}_n \circ \mathcal{I}_n \circ \mathcal{L}_n, \qquad n \in \mathbb{N}, \tag{3.28}$$

where $\mathcal{U}_n$ is the full elimination of the modes in I_n^- as in Theorem 3.4 (for $t = 1$). The *n-th step renormalization* operator is thus

$$\mathcal{R}_n = \mathcal{N}_n \circ \dots \circ \mathcal{N}_1, \qquad n \in \mathbb{N},$$

which is analytic in its domain. Notice that $\mathcal{N}_n(\boldsymbol{\omega}^{(n-1)} + \boldsymbol{v}) = \boldsymbol{\omega}^{(n)}$, for every $\boldsymbol{v} \in \mathbb{C}^2$. Also, in case a vector field X is real-analytic, the same is true for $\mathcal{N}_n(X)$ and $\mathcal{R}_n(X)$.

3.11. **Trivial limit of renormalization.**

Theorem 3.5. *Let $\alpha \in BC$ and $\rho_0 > \mathcal{B}(\alpha)$. If $X \in \mathbb{I}_0^+ \mathcal{A}_{\rho_0}$ has winding ratio α^{-1}, then, for all $n \in \mathbb{N}$, X is in the domain of $\mathcal{R}_n$ and*

$$\|\mathcal{R}_n(X) - \mathcal{R}_n(\boldsymbol{\omega})\|_{\rho_n} \le \Theta_n \|X - \boldsymbol{\omega}\|_{\rho_0}, \tag{3.29}$$

where $\Theta_n = \beta_n^4 \prod_{i=0}^n \beta_i^2$.

Proof. If $n = 1$,

$$\|\mathcal{I}_1 \mathcal{L}_1(X) - \boldsymbol{\omega}^{(1)}\|'_{\rho_1+\nu} \le \|\mathcal{I}_1(\mathbb{I} - \mathbb{E})\mathcal{L}_1(X)\|'_{\rho_1+\nu} + \|\widehat{\mathcal{L}}_1(X)\|$$

and, using Lemma 3.2, $\|\widehat{\mathcal{L}}_1(X)\| \le \|\mathcal{I}_1(\mathbb{I} - \mathbb{E})\mathcal{L}_1(X)\|'_{\rho_1+\nu}$. So, from Proposition 3.3,

$$\|\mathcal{I}_1(\mathbb{I} - \mathbb{E})\mathcal{L}_1(X)\|'_{\rho_1+\nu} \le \phi_1^{-1} \|(\mathbb{I} - \mathbb{E})\mathcal{L}_1(X)\|'_{\rho_1+\nu+\log(\phi_1)} \tag{3.30}$$

where $\phi_1 = \|T^{(1)}\|/(C\sigma_1^2\alpha)$. Now, Proposition 3.1 yields that

$$\|(\mathbb{I} - \mathbb{E})\mathcal{L}_1(X)\|'_{\rho_1+\nu+\log(\phi_1)} \le 2\left(1 + \frac{2\pi}{\mathrm{e}\delta}\right) \frac{\|T^{(1)}\|}{\alpha} \|(\mathbb{I} - \mathbb{E})X\|_{(\phi_1+\delta)A_0}.$$

The estimate (3.30) and a sufficiently small choice of $C > 0$ (depending only on δ and ν which are fixed) guarantees $\mathcal{I}_1\mathcal{L}_1(X)$ to be in the domain of $\mathcal{U}_1$. By (3.20),

$$\|\mathcal{U}_1\mathcal{I}_1\mathcal{L}_1(X) - \boldsymbol{\omega}^{(1)}\|_{\rho_1} \le 2\|\mathcal{I}_1\mathcal{L}_1(X) - \boldsymbol{\omega}^{(1)}\|'_{\rho_1+\nu}.$$

Since $(\phi_1 + \delta)A_0 = \rho_0$ and $\|(\mathbb{I} - \mathbb{E})X\|_{\rho_0} \le \|X - \boldsymbol{\omega}\|_{\rho_0}$, we then get

$$\|\mathcal{R}_1(X) - \mathcal{R}_1(\boldsymbol{\omega})\|_{\rho_1} \le 8C\left(1 + \frac{2\pi}{\mathrm{e}\delta}\right) \frac{\alpha_1^2\beta_1^2\|\boldsymbol{\Omega}^{(2)}\|^2}{\|T^{(2)^{-1}}\|^2\|\boldsymbol{\Omega}^{(1)}\|^2} \|X - \boldsymbol{\omega}\|_{\rho_0}$$

proving (3.29) for $n = 1$ with a choice of a constant C.

For $n \in \mathbb{N}$, suppose that (3.29) is satisfied for $n-1$ and denote $X_{n-1} = \mathcal{R}_{n-1}(X) \in \mathbb{I}_{n-1}^+\mathcal{A}_{\rho_{n-1}}$. Then, similarly to the above case, (3.11) together with Propositions 3.1, 3.2 and 3.3, can be used to estimate $\mathcal{I}_n\mathcal{L}_n(X_{n-1}) - \boldsymbol{\omega}^{(n)}$ of the order of Θ_n, which implies that it is inside the domain of $\mathcal{U}_n$ (using an appropriate choice of the constant C). It remains to use (3.20). ☐

We can generalise the above result for X sufficiently close to $\boldsymbol{\omega}$ in $\mathcal{A}_{\rho_0+\nu}$ and not necessarily with only resonant modes, by using an initial operator $\mathcal{U}_0$ and applying Theorem 3.5 to $\mathcal{U}_0(X)$.

3.12. **Small strips.** We recover the large strip case by using an initial transformation $X_N = \mathcal{U}_N\mathcal{L}_n \dots \mathcal{U}_1\mathcal{L}_1(X)$ so that $X_N \in \mathbb{I}_N^+\mathcal{A}_{\rho_N}$ with

$$\rho_N = \frac{1}{B_{N-1}}\left[\rho_0 - \sum_{i=1}^N B_{i-1}(\nu + \delta)\right],$$

and a fixed choice of δ and ν such that $\rho_N > 0$ for $N \in \mathbb{N}$. For that consider a large enough N and $X - \boldsymbol{\omega}$ sufficiently small such that X_N verifies the conditions of Theorem 3.5, i.e. $\rho_N = \mathcal{O}(\beta_{N-1}^{-1})$ gives a large strip. We need to check that we can find N such that $\rho_N > \mathcal{B}(\alpha_N)$. This follows from

$$\mathcal{B}(\alpha_N) = \frac{1}{B_{N-1}}\left[\mathcal{B}(\alpha) - \mathcal{B}_N(\alpha) - \sum_{i=N+1}^{+\infty} B_{i-1}\log\left(\beta_{N-1}^{-2}\right)\right],$$

where $\mathcal{B}_N(\alpha)$ is the sum of the first N terms of $\mathcal{B}(\alpha)$. Notice that $\lim_{N\to+\infty}\mathcal{B}_N(\alpha) = \mathcal{B}(\alpha)$ and $\mathcal{B}(\alpha_N) > 0$ for all N. Thus, $\rho_N > \mathcal{B}(\alpha_N)$ for N large enough.

4. Analytic conjugacy to linear flow

As a consequence of Theorem 3.5, we obtain an analytic conjugacy between the flow generated by X and the linear flow, thus proving Theorem 1.1.

Consider the set $\Delta_\alpha \subset \mathbb{I}_0^+\mathcal{A}_{\rho_0}$ inside the domain of $\mathcal{R}_n$ for all $n \in \mathbb{N}$, whose elements have winding ratio α^{-1}. By taking $X \in \Delta_\alpha$, we denote $X_n = \mathcal{R}_n(X) \in \mathbb{I}_n^+\mathcal{A}_{\rho_n}$ so that

$$X_n = \xi_n(X)\,(L_1 \circ U_1 \cdots L_n \circ U_n)^*(X), \tag{4.1}$$

where $U_k = U_k(X) = \mathfrak{U}_k(\mathcal{I}_k\mathcal{L}_k(X_{k-1}))$ is given by the analytic map in Theorem 3.4 for $t = 1$ at the k-th step, and

$$\xi_n(X) = \tau_n(\mathbb{E}X_{n-1}) \dots \tau_1(\mathbb{E}X). \tag{4.2}$$

Notice that if $X_n = \boldsymbol{\omega}^{(n)}$ for some $n \in \mathbb{N}$, X is analytically conjugated to $\boldsymbol{\omega}^{(n)}$.

Now, for each X, define the isotopic to the identity analytic diffeomorphism

$$W_n(X) = {P^{(n)}}^{-1} \circ U_n(X) \circ P^{(n)}, \tag{4.3}$$

on ${P^{(n)}}^{-1}\mathcal{D}_{\rho_n}$. If X is real-analytic, then $W_n(X)(\mathbb{R}^2) \subset \mathbb{R}^2$, since this property holds for $U_n(X)$. We also have $W_n(\boldsymbol{\omega}) = \mathrm{Id}$.

Take a sequence $R_n > 0$ such that $R_n\|P^{(n)}\| \leq \rho_n$, there exist constants R, R' satisfying

$$0 < R \leq R_n \leq R' < +\infty$$

and, for some $0 < c_1 < 1$ and $c_2 > 0$,

$$R_{n-1} - R_n > c_2\Theta_n^{c_1}. \tag{4.4}$$

Lemma 4.1. *There is an open ball $B \subset \Delta_\alpha$ about $\boldsymbol{\omega}$ such that, for all $n \in \mathbb{N}$, $W_n \colon B \to \mathcal{A}_{R_n}$ is analytic, satisfies $W_n(X) \colon \mathcal{D}_{R_n} \to \mathcal{D}_{R_{n-1}}$ and*

$$\|W_n(X) - \mathrm{Id}\,\|_{R_n} \leq c\Theta_n^{c'}\|X - \boldsymbol{\omega}\|_\rho, \quad X \in B, \tag{4.5}$$

with some constants $c, c' > 0$.

Proof. For any $X \in \Delta_\alpha$, in view of (3.20), we get

$$\begin{aligned}\|W_n(X) - \mathrm{Id}\,\|_{R_n} &= \|{P^{(n)}}^{-1} \circ [U_n(X) - \mathrm{Id}] \circ P^{(n)}\|_{R_n} \\ &\ll \sigma_n^{-1}\|{P^{(n)}}^{-1}\|\,\|\mathcal{I}_n\mathcal{L}_n(X_{n-1}) - \boldsymbol{\omega}^{(n)}\|_{\rho_n}.\end{aligned}$$

We can bound the above as in (4.5) for some $c, c' > 0$.

We shall choose a small enough open ball B about $\boldsymbol{\omega}$ in Δ_α, such that, for all $n \in \mathbb{N}$ and appropriate choices of c_1 and c_2 with $c' > c_1$,

$$\|X - \boldsymbol{\omega}\|_\rho < \frac{R_{n-1} - R_n}{2\pi c\Theta_n^{c'}}, \qquad X \in B.$$

Therefore, for $\boldsymbol{x} \in \mathcal{D}_{R_n}$ and $X \in B$,

$$\begin{aligned}\|\operatorname{Im} W_n(X)(\boldsymbol{x})\| &\leq \|\operatorname{Im}(W_n(X)(\boldsymbol{x}) - \boldsymbol{x})\| + \|\operatorname{Im} \boldsymbol{x}\| \\ &< \|W_n(X) - \mathrm{Id}\,\|_{R_n} + R_n/2\pi < R_{n-1}/2\pi.\end{aligned}$$

So we have $W_n(X) \colon \mathcal{D}_{R_n} \to \mathcal{D}_{R_{n-1}}$ and $W_n(X) \in \mathcal{A}_{R_n}$. From the properties of $\mathfrak{U}_n$, W_n is analytic. $\square$

Consider the analytic map $H_n\colon B \to \mathcal{A}_{R_n}$ defined by the coordinate transformation $H_n(X)\colon \mathcal{D}_{R_n} \to \mathcal{D}_{\rho_0}$ as

$$H_n(X) = W_1(X) \circ \cdots \circ W_n(X). \tag{4.6}$$

In addition, take the analytic map $\eta_n\colon B \to \mathbb{C}$ given by

$$\eta_n(X) = \beta_{n-1}\xi_n(X). \tag{4.7}$$

Lemma 4.2. *There exists $c, c' > 0$ such that for $X \in B$ and $n > 1$,*

$$\begin{aligned} \|H_n(X) - H_{n-1}(X)\|_{R_n} &\le c\Theta_n^{c'}\|X - \boldsymbol{\omega}\|_\rho, \\ |\eta_n(X) - \eta_{n-1}(X)| &\le c\beta_{n-1}. \end{aligned} \tag{4.8}$$

Proof. For each $k = 1, \dots, n-1$, consider the transformations

$$\begin{aligned} G_k(z, X) =&(W_k(X) - \mathrm{Id}) \circ (\mathrm{Id} + G_{k+1}(z, X)) + G_{k+1}(z, X), \\ G_n(z, X) =&z(W_n(X) - \mathrm{Id}), \end{aligned}$$

with $(z, X) \in \{z \in \mathbb{C}\colon |z| < 1 + d_n\} \times B$, where we have constants $c', c'' > 0$ such that

$$d_n = \frac{c''}{\Theta_n^{c'}\|X - \boldsymbol{\omega}\|_\rho} - 1 > 0.$$

If the image of $\mathcal{D}_{R_n}$ under $\mathrm{Id} + G_{k+1}(z, X)$ is inside the domain of $W_k(X)$, or simply

$$\|G_{k+1}(z, X)\|_{R_n} \le (R_k - R_n)/2\pi,$$

then G_k is well-defined as an analytic map into $\mathcal{A}_{R_n}$, and

$$\|G_k(z, X)\|_{R_n} \le \|W_k(X) - \mathrm{Id}\|_{R_k} + \|G_{k+1}(z, X)\|_{R_n}.$$

An inductive scheme shows that

$$\begin{aligned} \|G_n(z, X)\|_{R_n} \le&(R_{n-1} - R_n)/2\pi, \\ \|G_k(z, X)\|_{R_n} \le&\sum_{i=k}^{n-1} \|W_i(X) - \mathrm{Id}\|_{R_i} + |z|\,\|W_n(X) - \mathrm{Id}\|_{R_n} \\ \le&(R_{k-1} - R_n)/2\pi. \end{aligned}$$

By Cauchy's formula

$$\begin{aligned} \|H_n(X) - H_{n-1}(X)\|_{R_n} &= \|G_1(1, X) - G_1(0, X)\|_{R_n} \\ &= \left\| \frac{1}{2\pi i} \oint_{|z|=1+d_n/2} \frac{G_1(z, X)}{z(z-1)} dz \right\|_{R_n}, \end{aligned}$$

and

$$\begin{aligned} \|H_n(X) - H_{n-1}(X)\|_{R_n} &\le \frac{2}{d_n} \sup_{|z|=1+d_n/2} \|G_1(z, X)\|_{R_n} \\ &\ll \Theta_n^{c'}\|X - \boldsymbol{\omega}\|_\rho. \end{aligned}$$

Finally, we have

$$\begin{aligned} |\eta_n(X) - \eta_{n-1}(X)| &= \beta_{n-2}|\alpha_{n-1}\tau_n(\mathbb{E}X_{n-1}) - 1|\,|\xi_{n-1}(X)| \\ &\le \beta_{n-1} \prod_{i=1}^{n} (1 + \beta_{i-1}) \ll \beta_{n-1}. \end{aligned} \tag{4.9}$$

□

Denote by Diff_{per} the set of $\mathbb{Z}^2$-periodic diffeomorphisms and consider the same norm as $\|\cdot\|_r$.

Lemma 4.3. *There exist an open ball $B' \subset B$ about $\boldsymbol{\omega}$, $H\colon B' \to \operatorname{Diff}_{per}(\mathcal{D}_R, \mathbb{C}^2)$ and $\eta\colon B' \to \mathbb{C}$ such that for $X \in B'$, $H(X) = \lim_{n\to+\infty} H_n(X)$, $\eta(X) = \lim_{n\to+\infty} \eta_n(X)$ and*

$$\|H(X) - \operatorname{Id}\|_R \leq c\|X - \boldsymbol{\omega}\|_\rho, \quad |\eta(X) - 1| \leq c\|X - \boldsymbol{\omega}\|_\rho, \tag{4.10}$$

for some $c > 0$. If $X \in B'$ is real-analytic, then $H(X) \in \operatorname{Diff}^\omega_{per}(\mathbb{R}^2, \mathbb{R}^2)$ and $\eta(X) \in \mathbb{R}$.

Proof. Lemma 4.2 implies the existence of the limit $H_n(X) \to H(X)$ as $n \to +\infty$, for each X in a sufficiently small ball B, in the space $\operatorname{Diff}_{per}(\mathcal{D}_R, \mathbb{C}^2)$ which is closed when restricting to sufficiently close to identity diffeomorphisms. Moreover, $\|H(X) - \operatorname{Id}\|_R \ll \|X - \boldsymbol{\omega}\|_\rho$. The same can be said about $\eta(X)$. The convergence of H_n and η_n is uniform in B' so H and η are analytic. The fact that, for real-analytic X, $H(X)$ and $\eta(X)$ take real values for real arguments, follows from the same property of each $W_n(X)$ and $\tau_n(\mathbb{E}X)$. $\square$

Lemma 4.4. *For every $X \in B'$, $\eta(X)\, H(X)^*(X) = \boldsymbol{\omega}$ on $\mathcal{D}_R$.*

Proof. For each $n \in \mathbb{N}$ the definition of $H_n(X)$ and (4.1) imply that

$$H_n(X)^*(X) = \xi_n(X)^{-1} P^{(n)^*}(X_n). \tag{4.11}$$

Since $\xi_n(X)^{-1} P^{(n)^{-1}} \boldsymbol{\omega}^{(n)} = \eta_n(X)^{-1}\boldsymbol{\omega}$, the r.h.s. of (4.11) can be written as

$$\eta_n(X)^{-1}\boldsymbol{\omega} + \xi_n(X)^{-1} P^{(n)^*}(X_n - \boldsymbol{\omega}^{(n)}). \tag{4.12}$$

The second term can be estimated by

$$\begin{aligned} \sup_{\boldsymbol{x}\in\mathcal{D}_R} \|\xi_n(X)^{-1} P^{(n)^*}(X_n - \boldsymbol{\omega}^{(n)})(\boldsymbol{x})\| &\leq |\xi_n(X)^{-1}|\, \|P^{(n)^{-1}}\|\, \|X_n - \boldsymbol{\omega}^{(n)}\|_{\rho_n} \\ &\ll \Theta_n \|X - \boldsymbol{\omega}\|_\rho. \end{aligned} \tag{4.13}$$

Using the convergence of H_n and η_n, we complete the proof. $\square$

Appendix A. Elimination of modes

A.1. **Homotopy method.** In this section we prove Theorem 3.4 using a homotopy method. The proof is essentially the same of [14], we include it here for completeness.

As n is fixed, we will drop it from our notations. Also, write $\rho' = \rho_n$ and $\rho = \rho' + \nu$. First, we include a technical lemma that will be used below.

Lemma A.1. *Let $f \in \mathcal{A}'_\rho$. If $U = \operatorname{Id} + u$ where $u\colon \mathcal{D}_{\rho'} \to \mathcal{D}_{(\rho-\rho')/2}$ is in $\mathcal{A}_{\rho'}$ and $\|u\|_{\rho'} < (\rho - \rho')/4\pi$, then*

- $\|f \circ U\|_{\rho'} \leq \|f\|_{(\rho+\rho')/2}$,
- $\|Df \circ U\| \leq \|f\|'_{(\rho+\rho')/2}$,
- $\|f \circ U - f\|_{\rho'} \leq \|f\|'_{(\rho+\rho')/2}\, \|u\|_{\rho'}$,
- $\|Df \circ U - Df\| \leq \frac{4\pi}{\rho-\rho'} \|f\|'_\rho\, \|u\|_{\rho'}$.

The proof of these inequalities is straightforward and thus will be omitted. Now, assume that

$$\delta = 42\varepsilon/\sigma < 1/2.$$

For vector fields in the form $X = \boldsymbol{\omega} + f$, consider f to be in the open ball $\mathcal{E}$ in $\mathcal{A}'_\rho$ centred at the origin with radius ε. The coordinate transformation U is written as $U = \operatorname{Id} + u$, with u in

$$\mathcal{B} = \left\{u \in \mathbb{I}^- \mathcal{A}'_{\rho'} \colon u\colon \mathcal{D}_{\rho'} \to \mathcal{D}_\rho, \|u\|'_{\rho'} < \delta\right\}.$$

Notice that we have

$$\begin{aligned} \mathbb{I}^- U^*(X) &= \mathbb{I}^-(DU)^{-1}(\boldsymbol{\omega} + f \circ U) \\ &= (\mathbb{I}^-(I + Du)^{-1}(\boldsymbol{\omega} + f \circ U). \end{aligned}$$

Define the operator $F \colon \mathcal{B} \to \mathbb{I}^- \mathcal{A}_{\rho'}$,

$$F(u) = \mathbb{I}^-(I + Du)^{-1}(\boldsymbol{\omega} + f \circ U). \tag{A.1}$$

$F(u)$ takes real values for real arguments whenever u has that property. It is easy to see that the derivative of F at u is the linear map from $\mathbb{I}^- \mathcal{A}'_{\rho'}$ to $\mathbb{I}^- \mathcal{A}_{\rho'}$:

$$\begin{aligned} DF(u)\, h &= \mathbb{I}^-(I + Du)^{-1}[Df \circ U\, h \\ &\quad - Dh\, (I + Du)^{-1}\, (\boldsymbol{\omega} + f \circ U)]. \end{aligned} \tag{A.2}$$

We want to find a solution of

$$F(u_t) = (1 - t)F(u_0), \tag{A.3}$$

with $0 \le t \le 1$ and "initial" condition $u_0 = 0$. Differentiating the above equation with respect to t, we get

$$DF(u_t)\frac{du_t}{dt} = -F(0). \tag{A.4}$$

Proposition A.2. *If $u \in \mathcal{B}$, then $DF(u)^{-1}$ is a bounded linear operator from $\mathbb{I}^- \mathcal{A}_{\rho'}$ to $\mathbb{I}^- \mathcal{A}'_{\rho'}$ and*

$$\|DF(u)^{-1}\| < \delta/\varepsilon.$$

From the above proposition (to be proved in Section A.2) we integrate (A.4) with respect to t, obtaining the integral equation:

$$u_t = -\int_0^t DF(u_s)^{-1}\, F(0)\, ds. \tag{A.5}$$

In order to check that $u_t \in \mathcal{B}$ for any $0 \le t \le 1$, we estimate its norm:

$$\begin{aligned} \|u_t\|'_{\rho'} &\le t \sup_{v \in \mathcal{B}} \|DF(v)^{-1} F(0)\|'_{\rho'} \\ &\le t \sup_{v \in \mathcal{B}} \|DF(v)^{-1}\|\, \|\mathbb{I}^- f\|_{\rho'} < t\delta \|f\|_{\rho'}/\varepsilon, \end{aligned}$$

so, $\|u_t\|'_{\rho'} < \delta$. Therefore, the solution of (A.3) exists in $\mathcal{B}$ and is given by (A.5). Moreover, if X is real-analytic, then u_t takes real values for real arguments.

It is now easy to see that

$$U_t^*(X) - \boldsymbol{\omega} = \mathbb{I}^+ \sum_{n \ge 2} (-D(U_t - \mathrm{Id}))^n \boldsymbol{\omega} + \mathbb{I}^+ U_t^* f + (1 - t)\mathbb{I}^- f.$$

So, using Lemma A.1,

$$\begin{aligned} \|U_t^*(X) - \boldsymbol{\omega}\|_{\rho'} &\le \frac{1}{1 - \|u_t\|'_{\rho'}} (\|\boldsymbol{\omega}\|\, \|u_t\|'^{\,2}_{\rho'} + \|f\|_\rho) + (1 - t)\|f\|_{\rho'} \\ &< \frac{1}{1 - \delta} \left(\delta^2 \|\boldsymbol{\omega}\| \|f\|_{\rho'}/\varepsilon^2 + 1\right) \|f\|_\rho + (1 - t)\|f\|_{\rho'} \\ &< \left[\frac{1}{1 - \delta}\left(\frac{\delta^2 \|\boldsymbol{\omega}\|}{\varepsilon} + 1\right) + 1 - t\right] \|f\|'_\rho. \end{aligned}$$

Moreover, $\|U_t^*(X) - \boldsymbol{\omega} - \mathbb{I}^+ f - (1 - t)\mathbb{I}^- f\|_{\rho'} = \mathcal{O}(\|f\|_\rho^2)$, hence the derivative of $X \mapsto U_t^*(X)$ at $\boldsymbol{\omega}$ is $\mathbb{I} - t\mathbb{I}^-$.

A.2. **Proof of Proposition A.2.**

Lemma A.3. *If* $\|f\|'_\rho < \varepsilon < \sigma/4$, *then*

$$DF(0)^{-1} \colon \mathbb{I}^-\mathcal{A}_{\rho'} \to \mathbb{I}^-\mathcal{A}'_{\rho'}$$

is continuous and

$$\|DF(0)^{-1}\| < \frac{2}{\sigma - 4\|f\|'_\rho}.$$

Proof. From (A.2) one has

$$\begin{aligned} DF(0)\, h &= \mathbb{I}^-(\widehat{f} - D_{\boldsymbol{\omega}})\, h \\ &= -\left(\mathbb{I} - \mathbb{I}^- \widehat{f}\, D_{\boldsymbol{\omega}}^{-1}\right) D_{\boldsymbol{\omega}}\, h, \end{aligned}$$

where $\widehat{f}\, h = Df\, h - Dh\, f$ and $D_{\boldsymbol{\omega}}\, h = Dh\, \boldsymbol{\omega}$. Thus, the inverse of this operator, if it exists, is given by

$$DF(0)^{-1} = -D_{\boldsymbol{\omega}}^{-1}\left(\mathbb{I} - \mathbb{I}^- \widehat{f}\, D_{\boldsymbol{\omega}}^{-1}\right)^{-1}.$$

The inverse of $D_{\boldsymbol{\omega}}$ is the linear map from $\mathbb{I}^-\mathcal{A}_{\rho'}$ to $\mathbb{I}^-\mathcal{A}'_{\rho'}$:

$$D_{\boldsymbol{\omega}}^{-1}\, g(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I^-} \frac{g_{\boldsymbol{k}}}{2\pi\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\omega})} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

and is well-defined by the definition of I^-. So,

$$\begin{aligned} \|D_{\boldsymbol{\omega}}^{-1}\, g\|'_{\rho'} &< \sum_{\boldsymbol{k}\in I^-} \frac{1 + 2\pi\|\boldsymbol{k}\|}{2\pi\sigma\|\boldsymbol{k}\|} \|g_{\boldsymbol{k}}\| \mathrm{e}^{\rho'\|\boldsymbol{k}\|} \\ &\leq \frac{2}{\sigma}\|g\|_{\rho'}. \end{aligned}$$

Hence, $\|D_{\boldsymbol{\omega}}^{-1}\| < 2/\sigma$. It is possible to bound from above the norm of $\widehat{f} \colon \mathbb{I}^-\mathcal{A}'_{\rho'} \to \mathcal{A}_{\rho'}$ by $\|\widehat{f}\| \leq 2\|f\|'_{\rho'}$. Therefore,

$$\|\mathbb{I}^- \widehat{f}\, D_{\boldsymbol{\omega}}^{-1}\| < \frac{4}{\sigma}\|f\|'_{\rho'} < 1,$$

and

$$\left\| \left(\mathbb{I} - \mathbb{I}^- \widehat{f}\, D_{\boldsymbol{\omega}}^{-1}\right)^{-1} \right\| < \frac{\sigma}{\sigma - 4\|f\|'_{\rho'}}.$$

The statement of the lemma is now immediate. □

Lemma A.4. *Given* $u \in \mathcal{B}$, *the linear operator* $DF(u) - DF(0)$ *mapping* $\mathbb{I}^-\mathcal{A}'_{\rho'}$ *into* $\mathbb{I}^-\mathcal{A}_{\rho'}$, *is bounded and*

$$\|DF(u) - DF(0)\| < \frac{\|u\|'_{\rho'}}{1 - \|u\|'_{\rho'}} \left[\left(\frac{4\pi}{\rho - \rho'} + \frac{4 - 2\|u\|'_{\rho'}}{1 - \|u\|'_{\rho'}} \right) \|f\|'_\rho + \frac{2 - \|u\|'_{\rho'}}{1 - \|u\|'_{\rho'}} \|\boldsymbol{\omega}\| \right].$$

Proof. The formula (A.2) gives

$$\begin{aligned} [DF(u) - DF(0)]\; h \;&=\; \mathbb{I}^-(I + Du)^{-1}\, [Df \circ U\, h - (I + Du)Df\, h \\ &\qquad -Dh\, (I + Du)^{-1}(\boldsymbol{\omega} + f) \circ U \\ &\qquad +(I + Du)Dh\, (\boldsymbol{\omega} + f)] \\ &=\; \mathbb{I}^-(I + Du)^{-1}\{A + B + C\}, \end{aligned}$$

where

$$\begin{aligned}A &= [Df\circ U - Df - Du\,Df]\;h\\ B &= Du\,Dh\,(\boldsymbol{\omega}+f)\\ C &= -Dh\,(I+Du)^{-1}\,[f\circ U - f - Du\,(\boldsymbol{\omega}+f)]\,.\end{aligned}$$

Using Lemma A.1,

$$\begin{aligned}\|A\|_{\rho'} &\leq \left(\frac{4\pi}{\rho-\rho'}\|f\|'_{\rho}\|u\|_{\rho'}+\|f\|'_{\rho'}\|u\|'_{\rho'}\right)\|h\|_{\rho'},\\ \|B\|_{\rho'} &\leq (\|\boldsymbol{\omega}\|+\|f\|_{\rho'})\,\|u\|'_{\rho'}\|h\|'_{\rho'},\\ \|C\|_{\rho'} &\leq \frac{1}{1-\|u\|'_{\rho'}}\left[\|f\|'_{(\rho+\rho')/2}\|u\|_{\rho'}+\|u\|'_{\rho'}\,(\|\boldsymbol{\omega}\|+\|f\|_{\rho'})\right]\|h\|'_{\rho'}.\end{aligned}$$

□

To conclude the proof of Proposition A.2, notice that

$$\begin{aligned}\|DF(u)^{-1}\| &\leq \left(\|DF(0)^{-1}\|^{-1}-\|DF(u)-DF(0)\|\right)^{-1}\\ &< \left\{\frac{\sigma}{2}-2\varepsilon-\frac{\delta}{1-\delta}\left[\left(\frac{4\pi}{\rho-\rho'}+\frac{4-2\delta}{1-\delta}\right)\varepsilon+\frac{2-\delta}{1-\delta}\|\boldsymbol{\omega}\|\right]\right\}^{-1}\\ &< \frac{\delta}{\varepsilon}.\end{aligned}$$

The last inequality is true if

$$\varepsilon<\delta\left[\frac{\sigma}{2}-\frac{2\delta}{(1-\delta)^2}\|\boldsymbol{\omega}\|\right]\left[1+2\delta+\frac{\delta^2}{1-\delta}\left(\frac{4\pi}{\rho-\rho'}+\frac{4-2\delta}{1-\delta}\right)\right]^{-1}$$

with a positive numerator N and denominator D in the r.h.s. This is so for our choices of ε and $\delta<\frac{1}{2}$, by observing that

$$\frac{2\delta}{(1-\delta)^2}\|\boldsymbol{\omega}\|<12\delta\|\boldsymbol{\omega}\|<\frac{\sigma}{4},$$

so that $N>\delta\sigma/4$, $D<7$ and finally $\varepsilon\leq\frac{\sigma^2}{42\|\boldsymbol{\omega}\|}<\frac{\sigma}{42}<N/D$.

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