Notes on the Translated Curve Theorem

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Abstract

These notes are based on part of the series of lectures called "Introduction to Hamiltonian Dynamics and Invariant Curves of Twist Maps (K.A.M.)", given at the Workshop on Dynamical Systems held at the I.C.T.P. Trieste, 31st August to 18th September 1998, by Michael Herman (Université de Paris VII). Here we present the Theorem of the translated curve for C^k-perturbations, $k \geq 4$, as proved by Herman. For a complete exposition of the subject, including a proof for $k \geq 3$, see [1] and [2].

1 Translated curve Theorem

Definition 1 Let $\gamma > 0$ and $\tau \geq 1$. The (γ, τ)-Diophantine set is

$$DC_{\gamma,\tau} = \left\{ \alpha \in \mathbb{R}\colon \ |\alpha - p/q| \geq \gamma q^{-(\tau+1)}, p/q \in \mathbb{Q}, q \geq 1 \right\}.$$

$CT_\gamma \equiv DC_{\gamma,1}$ is the set of the constant type numbers.

Let $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ and $\mathbb{A}_\delta = \mathbb{T}^1 \times [-\delta, \delta]$, $\delta > 0$. For a function $\varphi\colon \mathbb{A}_\delta \to \mathbb{R}$ of class C^k, we consider the norm, for $k = 0$,

$$\|\varphi\|_{C^0} = \sup_{\mathbb{A}_\delta} |\varphi(\theta, r)|,$$

and, for $k \geq 1$, the semi-norm

$$\|\varphi\|_{C^k} = \sup_{1 \leq i+j \leq k} \sup_{\mathbb{A}_\delta} \left| \partial_\theta^i \partial_r^j \varphi(\theta, r) \right|.$$

If $\psi\colon \mathbb{T}^1 \to \mathbb{R}$ is a continuous mapping, in the same way we define the C^0-norm to be:

$$\|\psi\|_{C^0} = \sup_{\mathbb{T}^1} |\psi(\theta)|.$$

Definition 2 An L^2-Sobolev space is a set

$$W^{k,2}(\mathbb{T}^1) = \{\psi \in L^2(\mathbb{T}^1, \mathbb{R}; d\theta) \colon \|D^k\psi\|_{L^2} < +\infty\},$$

where $D^k\psi$ is the k-th derivative of ψ in the sense of the distributions, and the L^2-norm is given by $\|\cdot\|_{L^2} = \int_{\mathbb{T}^1} |\cdot|^2 \, d\theta$. That is, if $\psi(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\theta}$, then $D^k\psi(\theta) = \sum_{n\in\mathbb{Z}} (2\pi i n)^k a_n e^{2\pi i n\theta}$. This space, equiped with the norm

$$\|\psi\|_{W^{k,2}} = \left(|a_0|^2 + \|D^k\psi\|_{L^2}^2\right)^{\frac{1}{2}},$$

is a Hilbert space. Its topology is called the $W^{k,2}$-topology. We also define the related space $W_0^{k,2}(\mathbb{T}^1) = \{\psi \in W^{k,2}(\mathbb{T}^1) \colon \int_{\mathbb{T}^1} \psi(\theta) d\theta = 0\}$.

Note that $C^k(\mathbb{T}^1) \subset W^{k,2}(\mathbb{T}^1) \subset C^{k-\frac{1}{2}}(\mathbb{T}^1) \subset W^{k-1,2}(\mathbb{T}^1)$, and that the trigonometric polynomials are dense in $W^{k,2}(\mathbb{T}^1)$ for the $W^{k,2}$-topology.

Theorem 3 (Translated Curve Theorem) *Let $\delta > 0$, $\alpha \in CT_\gamma$ with*

$$\gamma = \inf_{p/q\in\mathbb{Q}} q^2|\alpha - p/q| > 0,$$

and an embedding $F \colon \mathbb{A}_\delta \to \mathbb{A}$ given by

$$F(\theta, r) = (\theta + \alpha + r + \varphi_1(\theta, r), r + \varphi_2(\theta, r)), \tag{1}$$

with $\varphi_j \in C^{k+1}(\mathbb{A}_\delta, \mathbb{R})$, $j \in \{1, 2\}$, $k \geq 3$. There are positive constants c_k and C_k such that, if

$$\sup_j \|\varphi_j\|_{C^{k+1}} \leq c_k \inf\{\gamma^2, \gamma\delta\},$$

then there is a unique pair $(\psi, \mu) \in W^{k,2}(\mathbb{T}^1) \times \mathbb{R}$ such that the map $f \colon \mathbb{T}^1 \to \mathbb{R}$,

$$f = \mathrm{Id} + \alpha + \psi + \varphi_1(\cdot, \psi)$$

has rotation number $\rho(f) = \alpha$, $\|\psi\|_{C^0} < \delta$ and

$$F(\theta, \psi(\theta)) = (f(\theta), \psi \circ f(\theta) + \mu), \tag{2}$$

for all $\theta \in \mathbb{T}^1$ (i.e. F translates the graph of ψ by μ). Furthermore,

$$\|\psi\|_{W^{k,2}} \leq \frac{C_k}{\gamma} \sup_j \|\varphi_j\|_{C^{k+1}}$$

and $|\mu| \leq \|\varphi_2\|_{C^0}$.

Remark 4 By a change of coordinates, we can set φ_1 to have a zero, i.e. $\varphi_1(\theta, r) = 0$ for some $(\theta, r) \in \mathbb{A}_\delta$. In this situation we can prove that there exists a constant $\kappa > 0$ such that

$$\|\varphi_1\|_{C^0} \leq \kappa \|\varphi_1\|_{C^{k+1}}. \tag{3}$$

This inequality will be used in the following.

2 Proof of Theorem 3

2.1 Preliminaries

For $k+1 \geq 3$, denote

$$D^k(\mathbb{T}^1) = \{f \in \mathrm{Diff}^k(\mathbb{R}) \colon f(x+1) = f(x)+1, x \in \mathbb{R}\}$$
$$D^{k,2}(\mathbb{T}^1) = \{f \in D^1(\mathbb{T}^1) \colon f - \mathrm{Id} \in W^{k,2}(\mathbb{T}^1)\}.$$

For some $\varepsilon_k > 0$ satisfying $\frac{\varepsilon_k \gamma}{2} < \frac{\delta}{4}$, consider the set

$$K^k_{\varepsilon_k,0} = \left\{\psi_0 \in W^{k,2}_0(\mathbb{T}^1) \colon \|D^k\psi_0\|_{L^2} \leq \frac{\varepsilon_k\gamma}{2}\right\}.$$

Having $\psi_0 \in K^k_{\varepsilon_k,0}$ let $\Psi_\alpha \colon \psi_0 \mapsto \psi = \psi_0 + \lambda_\alpha(\psi_0)$, where $\lambda_\alpha \colon K^k_{\varepsilon_k,0} \to \mathbb{R}$ is the function that gives the unique $\lambda_\alpha(\psi_0) \in \mathbb{R}$ such that $\rho(f_\psi) = \alpha$, with $f_\psi = \mathrm{Id} + \alpha + \psi + \varphi_1(\cdot,\psi)$ (see [3] II.). By this uniqueness property, it can be proved that the map $\psi_0 \mapsto \lambda_\alpha(\psi_0)$ is continuous for the C^0-topology.

There is the property that if $\rho(f) = \alpha$, then $f \circ R_{-\alpha}$ has a fixed point (see [3] II.). That is, there exists $\theta^* \in \mathbb{T}^1$ such that $f \circ R_{-\alpha}(\theta^*) = \theta^*$. Thus $|\lambda_\alpha(\psi_0)| = |\psi_0(\theta^*-\alpha) + \varphi_1(\theta^*-\alpha, \psi_0(\theta^*-\alpha) + \lambda_\alpha(\psi_0))|$, and

$$|\lambda_\alpha(\psi_0)| \leq \|\psi_0\|_{C^0} + \|\varphi_1\|_{C^0}.$$

If $\psi_0 = \sum_{n\neq 0} a_n e^{i2\pi n\theta} \in K^k_{\varepsilon_k,0}$, then

$$\begin{aligned}
\|\psi_0\|_{L^2} \leq \|\psi_0\|_{C^0} &\leq \sum_{n\neq 0} |a_n| = \sum_{n\neq 0} |a_n| \tfrac{n}{n} \\
&\leq \left(\sum_{n\neq 0} |a_n|^2 n^2 \sum_{n\neq 0} \tfrac{1}{n^2}\right)^{\frac{1}{2}} = \left(\tfrac{1}{(2\pi)^2} \sum_{n\neq 0} |a_n 2\pi i n|^2 \sum_{n>0} \tfrac{2}{n^2}\right)^{\frac{1}{2}} \\
&\leq \|D\psi_0\|_{L^2},
\end{aligned}$$

where we used the Cauchy-Schwartz inequality. By recurrence,

$$\|\psi_0\|_{C^0} \leq \|D^k\psi_0\|_{L^2} \leq \frac{\varepsilon_k\gamma}{2} < \frac{\delta}{4}. \tag{4}$$

The mapping $\psi_0 \mapsto f = \mathrm{Id} + \alpha + \psi_0 + \lambda_\alpha(\psi_0) + \varphi_1(\cdot, \psi_0 + \lambda_\alpha(\psi_0))$ is a bijection if $\|D^k f\|_{L^2} \leq b$, with b small. It is also weakly-continuous, i.e. continuous for the weak-topology, in $W^{k,2}(\mathbb{T}^1)$: If $\psi_0^{(n)} \rightharpoonup \psi_0$ (i.e. $\langle \psi_0^{(n)} - \psi_0, \xi\rangle \to 0$ as $n \to +\infty$ for all $\xi \in W^{k,2}(\mathbb{T}^1)$), then

$$\psi^{(n)} := \Psi_\alpha(\psi_0^{(n)}) \rightharpoonup \psi = \Psi_\alpha(\psi_0),$$

and

$$f_\psi^{(n)} := \mathrm{Id} + \alpha + \psi^{(n)} + \varphi_1(\cdot, \psi^{(n)}) \rightharpoonup f_\psi.$$

With the weak-topology the map $\Psi_\alpha \colon \psi_0 \mapsto \psi$ is an homeomorphism.

2.2 Functional equation

Let $G_\psi \colon \mathbb{T}^1 \to \mathbb{A}_\delta$ such that $G_\psi(\theta) = (\theta, \psi(\theta))$. Comparing (1) with (2), we want to solve the equation for (ψ, μ):

$$\psi \circ f_\psi - \psi = \varphi_2 \circ G_\psi + \mu. \tag{5}$$

Differentiating it three times, we get

$$D^3\psi \circ f_\psi (Df_\psi)^3 - D^3\psi[1 - D\psi \circ f_\psi(1 + \varphi_{1,r} \circ G_\psi) + \varphi_{2,r} \circ G_\psi] = B_\psi,$$

where

$$\begin{aligned} B_\psi = \; & D\left[\varphi_{2,\theta,\theta} \circ G_\psi + 2\varphi_{2,\theta,r} \circ G_\psi D\psi + \varphi_{2,r,r} \circ G_\psi (D\psi)^2\right] - 2D^2\psi \circ f_\psi Df_\psi D^2 f_\psi + \\ & + D^2\psi D[-D\psi \circ f_\psi(1 + \varphi_{1,r} \circ G_\psi) + \varphi_{2,r} \circ G_\psi] - \\ & - D\{D\psi \circ f_\psi[D(\varphi_{1,\theta} \circ G_\psi) + D(\varphi_{1,r} \circ G_\psi) D\psi]\}. \end{aligned}$$

When $\psi \in W^{k,2}(\mathbb{T}^1)$, as $F \in C^{k+1}(\mathbb{A}_\delta)$, we have that $B_\psi \in W^{k-2,2}(\mathbb{T}^1)$, $k \geq 3$. Divide the equation by $(Df_\psi)^3$ and denote

$$\begin{cases} a_\psi = (Df_\psi)^{-3}[1 - D\psi \circ f_\psi(1 + \varphi_{1,r} \circ G_\psi) + \varphi_{2,r} \circ G_\psi] \\ \eta_\psi = (Df_\psi)^{-3} B_\psi, \end{cases} \tag{6}$$

where $a_\psi \in W^{k-1,2}(\mathbb{T}^1)$ and $\eta_\psi \in W^{k-2,2}(\mathbb{T}^1)$. Then, we want to find a solution to

$$D^3\psi \circ f_\psi - a_\psi D^3\psi = \eta_\psi.$$

If ε_k and c_k are small enough, then

$$\|a_\psi - 1\|_{C^0} \leq \frac{1}{4}.$$

A good choice of c_k is the one given by

$$c_k \inf\{\gamma^2, \gamma\delta\} \leq C \frac{\varepsilon_k \gamma}{2},$$

for some constant $C > 0$ such that

$$\|f_\psi - R_\alpha\|_{W^{k,2}} \leq \varepsilon_k \gamma \quad \text{and} \quad \|\varphi_1\|_{C^0} \leq \varepsilon_k \gamma.$$

Note that $\|D^k f_\psi\|_{L^2} \leq \|f_\psi - R_\alpha\|_{W^{k,2}} \leq \varepsilon_k\gamma$. Using (3) and

$$\begin{aligned} \|f_\psi - R_\alpha\|^2_{W^{k,2}} & \leq (|\lambda_\alpha(\psi_0)| + \|\varphi_1\|_{C^0})^2 + (\|D^k\psi_0\|_{L^2} + \|\varphi_1\|_{C^{k+1}})^2 \\ & \leq \tfrac{\varepsilon_k\gamma}{2}\left[(1 + 2\kappa C)^2 + (1 + C)^2\right]^{\frac{1}{2}}, \end{aligned}$$

we find that it is possible to get $C > 0$ under the above conditions. This implies that, for any $\psi_0 \in K^k_{\varepsilon_k,0}$, the function $\psi = \Psi(\psi_0)$ verifies:

$$\|\psi\|_{C^0} \leq \|\psi_0\|_{C^0} + |\lambda_\alpha(\psi_0)| < \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta. \tag{7}$$

So, in the following we will be able to make use of the next result:

Lemma 5 *Let $f \in D^{k,2}(\mathbb{T}^1)$, $k \geq 3$, with $\rho(f) = \alpha$ and $\|f - R_\alpha\|_{W^{k,2}} \leq \varepsilon_k \gamma$. If $a \in W^{k-1,2}(\mathbb{T}^1)$ and $\|a-1\|_{C^0} \leq \frac{1}{4}$, for all $\eta \in W^{k-2,2}(\mathbb{T}^1)$, then there exists a unique pair $(\psi_1, \nu) \in W_0^{k-3,2}(\mathbb{T}^1) \times \mathbb{R}$ such that*

$$\psi_1 \circ f - a\psi_1 = \eta + \nu.$$

Furthermore,

$$\|D^{k-3}\psi_1\|_{L^2} \leq \frac{C'}{\gamma}\|D^{k-2}\eta\|_{L^2}$$

for some constant $C' > 0$ depending only on $\|D^{k-1}a\|_{L^2}$ and ε_k.

So, given a function $\psi_0 \in K^k_{\varepsilon_k,0}$, and thus f_ψ, a_ψ and η_ψ with $\psi = \Psi(\psi_0)$, we get a unique function ψ_{1,ψ_0} and a constant ν such that

$$\psi_{1,\psi_0} \circ f_\psi - a_\psi \psi_{1,\psi_0} = \eta_\psi + \nu, \tag{8}$$

and

$$\|D^{k-3}\psi_{1,\psi_0}\|_{L^2} \leq \ldots \leq \frac{\varepsilon_k \gamma}{2}.$$

2.3 The mapping Φ

The above result leads us to define the mapping $\Phi\colon K^k_{\varepsilon_k,0} \to K^k_{\varepsilon_k,0}$, such that

$$D^3(\Phi(\psi_0)) = \psi_{1,\psi_0}. \tag{9}$$

It is well defined (by integration within the space $K^k_{\varepsilon_k,0}$) since $\psi_{1,\psi_0} \in K^{k-3}_{\varepsilon_k,0}$ is unique for each $\psi_0 \in K^k_{\varepsilon_k,0}$.

Lemma 6 *Φ is weakly-continuous.*

Proof: Given a sequence $K^k_{\varepsilon_k,0} \ni \psi_0^{(n)} \rightharpoonup \psi_0 \in K^k_{\varepsilon_k,0}$, we have, with $\psi^{(n)} = \Psi_\alpha(\psi_0^{(n)}) \rightharpoonup \psi = \Psi_\alpha(\psi_0)$, $f_{\psi^{(n)}} \rightharpoonup f_\psi$, $a_{\psi^{(n)}} \rightharpoonup a_\psi$ and $\eta_{\psi^{(n)}} \rightharpoonup \eta_\psi$. Note that Ψ_α is an homeomorphism for the weak-topology. For all $n \in \mathbb{N}$ there is a function $\psi_1^{(n)}$ such that $\psi_1^{(n)} \circ f_{\psi^{(n)}} - a_{\psi^{(n)}}\psi_1^{(n)} = \eta_{\psi^{(n)}} + \nu^{(n)}$. Taking any convergent subsequence with limit $\bar{\psi}_1$, we get $\bar{\psi}_1 \circ f_\psi - a_\psi \bar{\psi}_1 = \eta_\psi + \bar{\nu}$. By uniqueness of the pair (ψ_1, ν) we have $\bar{\psi}_1 = \psi_1$ and $\bar{\nu} = \nu$. Thus $\psi_1^{(n)} \rightharpoonup \psi_1$, i.e. $D^3(\Phi(\psi_0^{(n)})) \rightharpoonup D^3(\Phi(\psi_0))$. By integration Φ is weakly-continuous. $\square$

2.4 Existence of solution

Lemma 7 (Schauder-Tychonoff fixed point) *Let E be a locally convex topological and Hausdorff vector space, and $K \subset E$ compact, convex. If $\Phi\colon K \to K$ is a continuous map, then Φ has a fixed point. In particular, $E = L^2$ with the weak topology and $K = K^k_{\varepsilon_k,0}$.*

So, there is some $\psi_0^* \in K^k_{\varepsilon_k,0}$ such that $\Phi(\psi_0^*) = \psi_0^*$. Let $\psi^* = \psi_0^* + \lambda_\alpha(\psi_0^*)$. Thus, $f_{\psi^*} \in D^{k,2}(\mathbb{T}^1)$ satisfies $\rho(f_{\psi^*}) = \alpha$. The graph of ψ^* is included in $\mathbb{A}_\delta$ as given by (7). Furthermore, from (8) and (9), $D^3\psi^* \circ f_{\psi^*} - a_{\psi^*} D^3\psi^* = \eta_{\psi^*} + \nu$, with a_{ψ^*} and η_{ψ^*} given by (6). This is the same as writing

$$D^3(\psi^* \circ f_{\psi^*} - \psi^* - \varphi_2 \circ G_{\psi^*}) = \nu (Df_{\psi^*})^3.$$

Because $Df_{\psi^*} > 0$ and the first term of the above equation is the derivative of a periodic function on $\mathbb{T}^1$ (meaning that it must have a zero), we have $\nu = 0$. Integrating this equation on $\mathbb{T}^1$, i.e. we are looking for 1-periodic solutions, we get

$$\psi^* \circ f_{\psi^*} - \psi^* = \varphi_2 \circ G_{\psi^*} + \mu, \tag{10}$$

for some constant $\mu \in \mathbb{R}$. Therefore, we found a pair-solution $(\psi^*, \mu) \in W^{k,2}(\mathbb{T}^1) \times \mathbb{R}$ for the equation (5).

2.5 Uniqueness of solution

Consider two pairs of solutions (ψ_1, μ_1) and (ψ_2, μ_2) of (5), and the curves C_1 and C_2 graphs of the functions ψ_1 and ψ_2 respectively. Uniqueness follows from the prove that these pair-solutions must be the same.

Let $F_1 = L_{-\mu_1} \circ F$ and $F_2 = L_{-\mu_2+\mu_1} \circ F$ be maps of $\mathbb{A}_\delta$ into itself, where $L_\mu(\theta, r) = (\theta, r + \mu)$ is a translation in r. Thus $F_i|_{C_i}$ is conjugated by the first projection to f_{ψ_i}. As $\rho(f_{\psi_1}) = \rho(f_{\psi_2}) = \alpha \notin \mathbb{Q}$, by [3] III.4 we have that $f_{\psi_1} \circ f_{\psi_2}^{-1}$ has a fixed point. Hence $C_1 \cap C_2 \neq \emptyset$ and

$$F_1(C_1 \cap C_2) \cap C_1 \neq \emptyset, \tag{11}$$

because $F_1(C_1) = C_1$.

Suppose that $\mu_1 \neq \mu_2$. Then $F_1(C_2) \cap C_2 = \emptyset$ and the open annulus $A \subset \mathbb{A}$, with borders C_2 and $F_1(C_2)$ is wandering by F_1, i.e. for any $n \geq 1$, $F_1^n(A) \cap A = \emptyset$. As C_1 is a F_1-invariant set and $F_1|_{C_1}$ is a minimal homeomorphism (the F_1-orbit of any point in C_1 is dense), we have $F_1(C_2) \cap C_1 = \emptyset$. Otherwise it should exist an integer $n \geq 1$ such that $F_1^n(A \cap C_1) \cap A = \emptyset$ contradicting the minimality of $F_1|_{C_1}$ (see Fig. 1). So, by (11), the hyphothesis $\mu_1 \neq \mu_2$ is absurd.

Now, as $F_1(C_i) = C_i$, the F_1-orbit of $(\theta, r) \in C_1 \cap C_2$ is dense in C_1 and in C_2 (by the minimality of F_1 in these sets). Therefore, $C_1 = C_2$, i.e. $\psi_1 = \psi_2$.

2.6 The inequalities

Here, we will make use of the local conjugacy Theorem for Sobolev spaces (see [1] pp.184 for a proof), since we have $\|D^k f_\psi\|_{L^2} \leq \|f_\psi - R_\alpha\|_{W^{k,2}} \leq \varepsilon_k \gamma$ and, with $f_\psi \in D^{2,2}(\mathbb{T}^1)$, Denjoy's theorem gives us that $\log Df_\psi$ is of bounded variation.

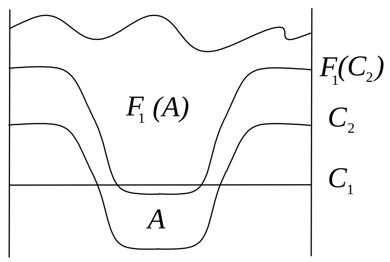

Figure 1: If the curve $F_1(C_2)$ intersects C_1, then there will be no iteration under F_1 of $A \cap C_1$ back in A (all the curves are in $\mathbb{A}$).

Lemma 8 (Local conjugacy) *There exists $\varepsilon_k > 0$, $C'' > 0$ such that, if $f \in D^{k,2}(\mathbb{T}^1)$, $\rho(f) = \alpha \in CT_\gamma$, $k \geq 3$, and*

$$\|\log Df\|_{C^0} \leq 1, \quad \|D^k f\|_{L^2} \leq \varepsilon_k \gamma,$$

then there is $h \in D^{k-1,2}(\mathbb{T}^1)$ with $h(0) = 0$, $f = h \circ R_\alpha \circ h^{-1}$ and

$$\|D^{k-1}h\|_{L^2} \leq \frac{C''}{\gamma}\|D^k f\|_{L^2} \leq C''\varepsilon_k.$$

By the previous Lemma $f_{\psi^*} = h \circ R_\alpha \circ h^{-1}$. Hence we rewrite (10):

$$\psi^* \circ h \circ R_\alpha - \psi^* \circ h = \varphi_2 \circ G_{\psi^*} \circ h + \mu.$$

Integrating over $\mathbb{T}^1$,

$$|\mu| = \left| \int_{\mathbb{T}^1} \varphi_2 \circ G_{\psi^*} \circ h(\theta) d\theta \right| \leq \sup_{\mathbb{A}_\delta} |\varphi_2(\theta, r)| = \|\varphi_2\|_{C^0}.$$

When ε_k is small,

$$\|D^3\psi_0^*\|_{W^{k-3,2}} \leq \frac{C'}{\gamma}\|\eta_{\psi^*}\|_{W^{k-2,2}} \leq \frac{C'''}{\gamma} \sup_j \|\varphi_j\|_{C^{k+1}},$$

for some constant $C''' > 0$. Thus, using (4),

$$\begin{aligned}
\|\psi^*\|^2_{W^{k,2}} &= |\lambda_\alpha(\psi_0^*)|^2 + \|D^k\psi_0^*\|^2_{L^2} \\
&\leq (\|D^3\psi_0^*\|_{W^{k-3,2}} + \kappa\|\varphi_1\|_{C^{k+1}})^2 + \|D^3\psi_0^*\|^2_{W^{k-3,2}} \\
&\leq \tfrac{C_k}{\gamma} \sup_j \|\varphi_j\|_{C^{k+1}},
\end{aligned}$$

for a constant $C_k > 0$. This completes the proof.

3 Proof of Lemma 5

Consider $\alpha \in CT_\gamma$.

Lemma 9 *Let $\eta \in W_0^{k,2}$, $k \geq 1$, and $\lambda > 0$. Then there exists a unique $\phi_\lambda \in W_0^{k-1,2}$ such that*

$$\phi_\lambda \circ R_\alpha - \lambda \phi_\lambda = \eta, \tag{12}$$

and

$$\|D^{k-1}\phi_\lambda\|_{L^2} \leq \frac{1}{4\pi\gamma}\|D^k \eta\|_{L^2}. \tag{13}$$

Proof: We start by decomposing η and ϕ_λ in Fourier series:

$$\eta = \sum_{n \in \mathbb{Z}\setminus\{0\}} \hat{\eta}_n e^{2\pi i n\theta}, \quad \phi_\lambda = \sum_{n \in \mathbb{Z}\setminus\{0\}} \hat{\phi}_n e^{2\pi i n\theta}.$$

The function ϕ_λ solves (12) if $(e^{2\pi i n\alpha} - \lambda)\hat{\phi}_n = \hat{\eta}_n$, defining uniquely $\hat{\psi}_n$ and hence ψ. Because

$$\begin{aligned} |e^{2\pi i n\alpha} - \lambda| &= [(\lambda-1)^2\cos^2(n\pi\alpha) + (1+\lambda)^2\sin^2(n\pi\alpha)]^{\frac{1}{2}} \\ &\geq (1+\lambda)|\sin(n\pi\alpha)| \geq \tfrac{2\gamma}{|n|}, \end{aligned}$$

we have that $|\hat{\phi}_n| \leq \frac{|n|}{2\gamma}|\hat{\eta}_n| = \frac{1}{4\pi\gamma}|\widehat{D\eta}_n|$. I.e. $\phi_\lambda \in W_0^{k-1,2}$ and (13). $\square$

Remark 10 If $\lambda \neq 1$, then $\phi_\lambda \in W_0^{k,2}$.

Lemma 11 *Let $\ell \in C^0(\mathbb{T}^1)$, $\|\ell - 1\|_{C^0} \leq \frac{1}{4}$ and $a \in W^{k-1,2}$, $\|a-1\|_{C^0} \leq \frac{1}{4}$. Then, for any $\eta \in W_0^{k-3,2}$, $k \geq 3$, there is an unique pair $(\psi, \nu) \in W^{k-1,2} \times \mathbb{R}$ such that $\langle \psi, \ell \rangle_{L^2} = 0$ and*

$$\psi \circ R_\alpha - a\psi = \eta + \nu. \tag{14}$$

Furthermore,

$$\|D^{k-3}\psi\|_{L^2} \leq \frac{C}{\gamma}\|D^{k-2}\eta\|_{L^2},$$

where C is a constant depending on k and on $\|D^{k+1}a\|_{L^2}$.

Proof: We can find $b \in W^{k,2}$ such that

$$\log(b \circ R_\alpha) - \log b = -\log a + \int_{\mathbb{T}^1} \log a.$$

Therefore $a = \lambda b/(b \circ R_\alpha)$ with $\lambda = e^{\int_{\mathbb{T}^1} \log a}$. We look for (ψ, ν) such that, multiplying (14) by b and writing $\nu = \nu_0 + (1-\lambda)\mu$,

$$(\psi b) \circ R_\alpha - \lambda(\psi b) = b \circ R_\alpha(\eta + \nu),$$

where $\nu_0 = -\frac{\int_{\mathbb{T}^1}(b \circ R_\alpha)\eta}{\int_{\mathbb{T}^1} b}$.

When $\lambda = 1$ we have $\nu = \nu_0$. Let $\psi = \frac{1}{b}(\phi_1 + \mu)$, with $\phi_1 \in W_0^{k-1,2}$, then the function ϕ_1 satisfies

$$\phi_1 \circ R_\alpha - \phi_1 = (b \circ R_\alpha)(\eta + \nu_0).$$

Note that $(b \circ R_\alpha)(\eta + \nu_0) \in W_0^{k,2}$, so, by Lemma 9, we get a unique function ϕ_1 as above. For a given ℓ we choose uniquely μ such that $\langle(\phi_1 + \mu)/b, \ell\rangle = 0$, i.e. $\mu = -\int_{\mathbb{T}^1}(\phi_1 \ell/b)/\int_{\mathbb{T}^1}(\ell/b)$. Therefore, there is a unique $\psi = \frac{1}{b}(\phi_1 + \mu)$.

When $\lambda \neq 1$, following Remark 10, we have a unique $\phi_\lambda \in W_0^{k,2}$ such that

$$\phi_\lambda \circ R_\alpha - \lambda\phi_\lambda = (b \circ R_\alpha)(\eta + \nu_0),$$

We also have a unique $\chi \in W^{k,2}$ such that

$$\chi \circ R_\alpha - \lambda\chi = (b \circ R_\alpha)(1 - \lambda),$$

$\chi > 0$ and $\min_{\mathbb{T}^1} b(\theta) \leq \chi \leq \max_{\mathbb{T}^1} b(\theta)$, given by

$$\chi = (1-\lambda)\sum_{j=0}^{+\infty} \lambda^j b \circ R_{-j\alpha}, \quad \chi = -(1-\lambda)\sum_{j=1}^{+\infty} \frac{1}{\lambda^j} b \circ R_{j\alpha},$$

for $\lambda < 1$ and $\lambda > 1$, respectively. Note that $\int_{\mathbb{T}^1} \chi = \int_{\mathbb{T}^1} b$. The general solution of (14) is $\psi_\mu = \frac{1}{b}(\phi_\lambda + \mu\chi)$, $\nu_\mu = \nu_0 + (1-\lambda)\mu$. As ℓ and χ are of constant sign, there is μ, unique, such that $\langle\psi_\mu, \ell\rangle = 0$. The inequality follows from Lemma 9. □

We suppose that we are under the conditions of Lemmas 5 and 8. So, we have $f = h \circ R_\alpha \circ h^{-1}$ and thus

$$(\psi_1 \circ h) \circ R_\alpha - (a \circ h)(\psi_1 \circ h) = (\eta \circ h) + \nu.$$

Then apply Lemma 11 to $a \circ h \in W^{k-1,2}$, $\eta \circ h \in W^{k-2,2}$, $\psi_1 \circ h$ and $\ell = Dh$, satisfying $\|\ell - 1\|_{C^0} \leq \frac{1}{4}$. Hence, there exists a unique solution to the above equation, $(\psi_1 \circ h, \nu)$, such that

$$\int_{\mathbb{T}^1} \psi_1 \circ h Dh \, d\theta = \int_{\mathbb{T}^1} \psi_1 d\theta = 0 \Rightarrow \psi_1 \in W_0^{k-3,2},$$

and

$$\|D^{k-3}\psi_1\|_{L^2} \leq \frac{cte}{\gamma}\|D^{k-2}(\eta \circ h)\|_{L^2} \leq \frac{C'}{\gamma}\|D^{k-2}\eta\|_{L^2}.$$

This concludes the proof.

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